

THE LAGUERRE INEQUALITY AND THE DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT. The purpose of this paper is fourfold: (1) to survey some classical and recent results in the theory of distribution of zeros of entire functions, (2) to generalize the Laguerre inequality, (3) to establish several special cases of the Hawai'i Conjecture, and (4) to present some new results dealing with the polar derivatives of polynomials. In addition, applications of the extended Laguerre inequalities are given. The paper concludes with several open problems.

0. INTRODUCTION

This paper is organized under the following section headings:

1. THE LAGUERRE-PÓLYA CLASS
2. THE LAGUERRE INEQUALITY
 - 2.1. The Extended Laguerre Expression
3. THE DISTRIBUTION OF ZEROS OF POLYNOMIALS
4. NECESSARY AND SUFFICIENT CONDITIONS
5. THE CENTER OF MASS WITH RESPECT TO A FINITE POINT
6. THE POLAR DERIVATIVE
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 - 10.1. Proof of Theorem 2.5 (Section 2.1)
 - 10.2. Calculation of the Polar Derivative Laguerre Expression
 - 10.3. Composition Theorems

In Section 1 we introduce the Laguerre-Pólya class, $\mathcal{L} - \mathcal{P}$, of real entire functions and recall several facts about this class of functions. The Laguerre inequality and the extended Laguerre inequalities are discussed in Section 2. Several important

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theorems in the theory of the distribution of zeros of complex polynomials are reviewed in Section 3. In Section 4 we consider some necessary and sufficient conditions for a function to belong to $\mathcal{L} - \mathcal{P}$. In addition, we establish a new condition for a real polynomial to have only real zeros. The center of mass of the zeros of a polynomial is the focal point of our study in Section 5. Here we also present a new proof of a special case of a theorem from Section 4. The highlight of this paper is Section 6 which contains several new results concerning the polar derivative of a polynomial, including a new necessary condition for a polynomial to be in $\mathcal{L} - \mathcal{P}$. The open problem known as the Hawai'i Conjecture is discussed in Section 7, together with what appear to be new proofs of special cases of this conjecture. Section 8 includes some applications of the Laguerre inequality to special functions. Finally, in Section 9 we state several open problems involving the Hawai'i Conjecture, a discrete Laguerre inequality, and the polar derivative Laguerre expression. The material covered in Sections 1 - 9 is supplemented by an appendix (Section 10).

1. THE LAGUERRE-PÓLYA CLASS

We begin by defining a general class of functions, $\mathfrak{S}(A)$, as follows. Let

$$S(A) = \{z \in \mathbb{C} : |\Im(z)| \leq A\};$$

i.e., $S(A)$ is the horizontal strip symmetric about the real axis with width $2A$.

Definition 1.1. ([6]). Let A be such that $0 \leq A < \infty$. We say that a function f is in the class $\mathfrak{S}(A)$ if f is of the form,

$$f(z) = Ce^{-\alpha z^2 + \beta z} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k}},$$

where $\alpha \geq 0$, $z_k \in S(A) \setminus \{0\}$, and $\sum_{k=1}^{\infty} \frac{1}{|z_k|^2} < \infty$. A real entire function is in the *Laguerre-Pólya class* if it can be expressed in the form,

$$\psi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) e^{-\frac{x}{a_k}}$$

where c, β , and a_k are real, $\alpha \geq 0$, $n \in \mathbb{Z}^+$ and $\sum a_k^{-2} < \infty$. We write $\psi \in \mathcal{L} - \mathcal{P}$. Note that $\mathcal{L} - \mathcal{P}$ is the same as the class $\mathfrak{S}(0)$. A function φ is said to be in $\mathcal{L} - \mathcal{P}^*$ if $\varphi(x) = \psi(x)p(x)$, where $\psi \in \mathcal{L} - \mathcal{P}$ and $p(x)$ is a real polynomial. Thus, $\varphi \in \mathcal{L} - \mathcal{P}^*$ if and only if $\varphi \in \mathfrak{S}(A)$ for some $A \geq 0$ and φ has at most finitely many nonreal zeros.

For the various properties of functions in the Laguerre-Pólya class we refer to [2] (and the references contained therein). Here, we single out merely some facts which we will use in the sequel concerning functions in the Laguerre-Pólya class.

Let

$$(1.1) \quad \varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

be a real entire function.

Fact 1.2. The real entire function $\varphi(x)$ is in the Laguerre-Pólya class if and only if φ can be uniformly approximated on disks about the origin by a sequence of

polynomials with only real zeros. (For a modern proof of this result see [12, Chapter 8]).

Fact 1.3. Thus, it follows from Fact 1.2 that the class $\mathcal{L} - \mathcal{P}$ is closed under differentiation; that is, if $\varphi \in \mathcal{L} - \mathcal{P}$, then $\varphi^{(n)} \in \mathcal{L} - \mathcal{P}$ for $n \geq 0$.

Fact 1.4. If $\varphi \in \mathcal{L} - \mathcal{P}$ (cf. 1.1), then the *Jensen polynomials*, $g_n(x)$, associated with φ ,

$$g_n(x) = \sum_{k=0}^n \binom{n}{k} \gamma_k x^k, \quad n = 0, 1, 2, \dots,$$

have only real zeros.

Fact 1.5. If $\varphi \in \mathcal{L} - \mathcal{P}$, then it follows from Fact 1.4 that the *Turán inequalities* hold; that is,

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 1, 2, 3, \dots$$

Moreover, two consecutive terms of the sequence $\{\gamma_j\}_{j=0}^\infty$ cannot be zero unless all subsequent or preceding terms are zero; that is, if $\gamma_k^2 + \gamma_{k+1}^2 = 0$, then $\gamma_j = 0$ for all $j \leq k$ or $\gamma_j = 0$ for all $j \geq k+1$. Furthermore, if $\gamma_k = 0$, but $\gamma_{k-1}\gamma_{k+1} \neq 0$, then $\gamma_{k-1}\gamma_{k+1} < 0$.

Fact 1.6. If $\varphi \in \mathcal{L} - \mathcal{P}$, then the *Laguerre inequality* holds; that is,

$$(1.2) \quad L[\varphi^{(p)}](x) = (\varphi^{(p)})^2 - \varphi^{(p-1)}(x)\varphi^{(p+1)}(x) \geq 0$$

for all $x \in \mathbb{R}$ and $p = 0, 1, 2, \dots$.

In the sequel, we will call $L[\varphi^{(p)}](x)$ the *Laguerre expression* for $\varphi^{(p)}(x)$. (For a proof of Fact 1.6 see Section 2 below).

2. THE LAGUERRE INEQUALITY

Consider a real polynomial

$$p(x) = c \cdot \prod_{k=1}^n (x - a_k), \quad (c, a_k \in \mathbb{R}).$$

Computing the logarithmic derivative we get,

$$\frac{p'(x)}{p(x)} = (\log p(x))' = \left(\log c + \sum_{k=1}^n \log(x - a_k) \right)' = \sum_{k=1}^n (x - a_k)^{-1}.$$

Thus, by taking the derivative of the previous expression

$$Q(x) = \left(\frac{p'(x)}{p(x)} \right)' = \frac{p(x)p''(x) - (p'(x))^2}{(p(x))^2} = - \left[\sum_{k=1}^n (x - a_k)^{-2} \right] < 0.$$

Therefore, we conclude that if $p(x)$ has only real zeros, then $Q(x)$ has no real zeros.

Remark 2.1. Notice that we now have the Laguerre inequality (1.2) for polynomials with only real zeros

$$L[p](x) = (p'(x))^2 - p(x)p''(x) = (p(x))^2 \left[\sum_{k=1}^n (x - a_k)^{-2} \right] \geq 0.$$

Moreover, if p has a real zero of multiplicity greater than 1, then equality occurs. If for example, α is a zero of p with multiplicity $m \geq 2$, then $p'(\alpha) = 0$ and $p(\alpha) = 0$.

Hence, $(p'(\alpha))^2 - p''(\alpha)p(\alpha) = 0$. Conversely, if $p \in \mathcal{L} - \mathcal{P}$ and $L[p](\alpha) = 0$, then α is a zero of p of multiplicity $m \geq 2$.

If $\varphi(x) \in \mathcal{L} - \mathcal{P}$, then

$$(2.1) \quad \varphi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{z_k}\right) e^{-\frac{x}{z_k}}$$

where c, β , and z_k are real, $\alpha \geq 0$, $n \in \mathbb{Z}^+$ and $\sum z_k^{-2} < \infty$. Formally, taking the derivative of the logarithmic derivative yields,

$$\left(\frac{\varphi'(x)}{\varphi(x)}\right)' = -\frac{n}{x^2} - 2\alpha + \sum_{k=1}^{\infty} \frac{1}{z_k} \left(\frac{-\frac{1}{z_k}}{(1 + \frac{x}{z_k})^2}\right) \leq 0,$$

and hence,

$$L[\varphi](x) = (\varphi'(x))^2 - \varphi''(x)\varphi(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Simple examples show that the Laguerre inequality is not a sufficient condition for a real entire function to belong to $\mathcal{L} - \mathcal{P}$. Indeed, if $p(x) = x(x^2 + a^2)$, $a \neq 0$, then $p(x) \notin \mathcal{L} - \mathcal{P}$, but $L[p](x) = a^4 + 3x^4 \geq 0 \quad \forall x \in \mathbb{R}$. On the other hand, a real entire function of order 2, having *only real zeros* need *not* satisfy the Laguerre inequality for all $x \in \mathbb{R}$, as the following example shows.

Example 2.2. *The real entire function $f(x) = e^{x^2} \cos x$, is of order 2 and has only real zeros, but it is not in the Laguerre-Pólya class because the coefficient of the term x^2 in the exponent is nonnegative (cf. Definition 1.1). A calculation shows that the Laguerre expression for this function is*

$$\begin{aligned} L[f](x) &= (f'(x))^2 - f''(x)f(x) = e^{2x^2}(\sin^2 x - \cos^2 x) \\ &= -e^{2x^2}(\cos^2 x - \sin^2 x) \\ &= -e^{2x^2}(\cos 2x). \end{aligned}$$

Thus we see that for this function the Laguerre inequality is not satisfied for all $x \in \mathbb{R}$.

2.1. The Extended Laguerre Expression. In [13] Patrick defined the *extended Laguerre expressions* to be,

$$L_k[\varphi^{(n)}](x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} \varphi^{(n+j)}(x) \varphi^{(n+2k-j)}(x), \quad k, n = 0, 1, 2, \dots$$

associated with a real entire function φ . The motivation for introducing these expressions is two-fold. If $\varphi(x)$ is a real entire function, then $L_k[\varphi](x) \geq 0 \quad \forall x \in \mathbb{R}$ and $k = 0, 1, 2, \dots$ is a necessary and sufficient condition for φ to belong to the Laguerre-Pólya class (cf. Theorem 2.3 and Theorem 4.1 below). In addition, the extended Laguerre inequalities provide a tool for establishing a number of inequalities for large classes of special functions (cf. Section 8).

Patrick's proof of the extended Laguerre inequalities involves some complicated calculations. The following shorter and simpler proof is an adaptation of a proof given in [3].

Theorem 2.3. (The Extended Laguerre Inequalities [3], [13]). *Let $\varphi(z) = \varphi(x + iy) \in \mathcal{L} - \mathcal{P}$. Then for any integer $n \geq 0$ and any $z = x + iy$*

$$|\varphi^{(n)}(x + iy)|^2 = \sum_{k=0}^{\infty} L_k[\varphi^{(n)}](x) y^{2k},$$

where

$$L_k[\varphi^{(n)}](x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} \varphi^{(n+j)}(x) \varphi^{(n+2k-j)}(x).$$

Furthermore,

$$L_k[\varphi^{(n)}](x) \geq 0 \quad \text{for } k = 0, 1, 2, \dots$$

Proof. Using Taylor's theorem for a fixed x we get,

$$h(y) := |\varphi(x + iy)|^2 = \varphi(x + iy) \varphi(x - iy) = \sum_{k=0}^{\infty} \frac{h^{(2k)}(0)}{(2k)!} y^{2k}.$$

Since h is an even function of y , there are no odd powers of y . Thus, by Leibniz's rule

$$\begin{aligned} h^{(2k)}(0) &= \sum_{j=0}^{2k} \binom{2k}{j} \varphi^{(j)}(x) (i)^j \varphi^{(2k-j)}(x) (-i)^{2k-j} \\ &= \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{k+j} \varphi^{(j)}(x) \varphi^{(2k-j)}(x) \\ &= (2k)! L_k[\varphi](x). \end{aligned}$$

If $\varphi \in \mathcal{L} - \mathcal{P}$, then

$$\begin{aligned} |\varphi(x + iy)|^2 &= \varphi(x + iy) \varphi(x - iy) \\ &= c^2 e^{-\alpha(x+iy)^2 + \beta(x+iy)} e^{-\alpha(x-iy)^2 + \beta(x-iy)} \\ &\quad (x + iy)^m (x - iy)^m \prod_{k=1}^{\infty} \left(1 - \frac{x + iy}{z_k}\right) \left(1 - \frac{x - iy}{z_k}\right) e^{\frac{x+iy}{z_k}} e^{\frac{x-iy}{z_k}} \\ (2.2) \quad &= c^2 e^{-2\alpha x^2 + 2\beta x} e^{2\alpha y^2} (x^2 + y^2)^m \prod_{k=1}^{\infty} \left(\left(1 - \frac{x}{z_k}\right)^2 + \frac{y^2}{z_k^2} \right) e^{\frac{2x}{z_k}}. \end{aligned}$$

Therefore,

$$|\varphi(x + iy)|^2 = \sum_{k=0}^{\infty} L_k[\varphi](x) y^{2k}$$

and by (2.2), $L_k[\varphi](x) \geq 0$ for all $x \in \mathbb{R}$.

In order to complete the proof of the theorem, we recall (Fact 1.3) that if $\varphi \in \mathcal{L} - \mathcal{P}$ then $\varphi' \in \mathcal{L} - \mathcal{P}$, and hence

$$L_k[\varphi^{(n)}](x) = \sum_{j=0}^{2k} \binom{2k}{j} \frac{(-1)^{k+j}}{(2k)!} \varphi^{(n+j)}(x) \varphi^{(n+2k-j)}(x).$$

□

Remark 2.4. For the reader's convenience, we have listed here the following explicit expressions for $L_n[\varphi](x)$ when $n = 0, 1, 2$:

$$\begin{aligned} L_0[\varphi](x) &= (\varphi(x))^2 \\ L_1[\varphi](x) &= (\varphi'(x))^2 - \varphi(x)\varphi''(x) \\ L_2[\varphi](x) &= \frac{1}{4}(\varphi''(x))^2 - \frac{1}{3}\varphi'(x)\varphi^{(3)}(x) + \frac{1}{12}\varphi(x)\varphi^{(4)}(x). \end{aligned}$$

Note, that when $n = 1$, $L_1[\varphi](x)$ is the Laguerre expression.

Our next result shows that for any given nonnegative integer n , there exists a function $f \in \mathcal{L} - \mathcal{P}^*$ such that the extended Laguerre inequalities, $L_k[f](x) \geq 0$ for all $x \in \mathbb{R}$ and $0 \leq k \leq n$, hold but fail for some integer $k \geq n + 1$. The significance of this result will become apparent in Section 4 (cf. Theorem 4.1). Since the proof of Theorem 2.5 is elementary, albeit involved, we have relegated it to the Appendix.

Theorem 2.5. *For each nonnegative integer n , there exists a function $g \in \mathcal{L} - \mathcal{P}^*$ such that $L_k[g](x) \geq 0$ for $0 \leq k \leq n$ and for all $x \in \mathbb{R}$, but $L_{n+1}[g](x) < 0$ for some $x \in \mathbb{R}$, where $L_k[g]$ denotes the extended Laguerre expression.*

Proof. See the Appendix (Section 10.1) for the proof.

3. THE DISTRIBUTION OF ZEROS OF POLYNOMIALS

This section highlights several fundamental theorems in the theory of the distribution of zeros of polynomials. In particular, our goal here is to prove (1) a beautiful geometric result known as Laguerre's Separation Theorem (Theorem 3.6) and (2) Walsh's Two Circle Theorem (Theorem 3.8).

We remark that Laguerre's Separation Theorem plays a pivotal role in this investigation (see Section 6). The connection between Laguerre's Separation Theorem and the algebraic characterization of functions in the Laguerre-Pólya class is by no means obvious. The key result in this characterization rests on the Malo-Schur-Szegő Composition Theorem (cf. Appendix, Theorem 10.9). In the Appendix we show how Laguerre's Separation Theorem implies Grace's Apolarity Theorem which in turn yields the aforementioned composition theorem.

Theorem 3.1. (Gauss-Lucas Theorem [18, Chapter 3]). *If $p(z)$ is a complex polynomial, then all the zeros of $p'(z)$ are located in the closed convex hull of the zeros of $p(z)$.*

The proof of the Gauss-Lucas Theorem follows from the following lemma and the fact that a convex hull is the intersection of half planes.

Lemma 3.2. *If $p(z)$ is a complex polynomial with all of its zeros in a half plane, then $p'(z)$ has all of its zeros in the same half plane.*

Proof. By means of the transformation $z \mapsto az + b$, we can map any half plane to any other half plane. Thus, it suffices to prove the lemma for the right and left half planes, $H_l = \{z : \Re(z) < 0\}$ and $H_r = \{z : \Re(z) > 0\}$, respectively. Suppose that $p(z) = c \prod_{k=1}^n (z - a_k)$, $a_k \in \overline{H}_l$ for $k = 1, \dots, n$, and (temporarily) all the zeros are simple. Consider the logarithmic derivative,

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^n \frac{1}{z - a_k}.$$

Now, for any $z \in H_r$ we have, $\Re(z - a_k) = \Re(z) - \Re(a_k) > 0$, since $a_k \in \overline{H}_l$. Hence, $\Re(\frac{1}{z - a_k}) > 0$ and $\frac{1}{z - a_k} \in H_r$ for each $k = 1, \dots, n$. Therefore,

$$\sum_{k=1}^n \frac{1}{z - a_k} \neq 0$$

and if $z \in H_r$, then z cannot be a zero of $p'(z)$.

If $p(z)$ has some multiple zeros, then they are zeros of both p and p' , and hence they will lie in the same half plane. \square

Definition 3.3. If $z = \alpha + i\beta$ ($\beta \neq 0$) is a zero of a real entire function $f(z)$, then the *Jensen circle* of f is the circle centered at α with radius $|\beta|$.

Theorem 3.4. (Jensen's Theorem [1, p. 19]). *If $p(z)$ is a real polynomial, then the nonreal zeros of $p'(z)$ lie on or in some Jensen circle of $p(z)$.*

Proof. Let x_1, \dots, x_m denote the real zeros and let $z_k = \alpha_k + i\beta_k$, $\bar{z}_k = \alpha_k - i\beta_k$, $k = 1, \dots, d$ denote the nonreal zeros of p . Let $\hat{z} = \hat{x} + i\hat{y}$ denote a nonreal zero of $p'(z)$ which is not a zero of $p(z)$. Then,

$$\frac{p'(\hat{z})}{p(\hat{z})} = \sum_{j=1}^m \frac{1}{\hat{z} - x_j} + \sum_{k=1}^d \left[\frac{1}{\hat{z} - z_k} + \frac{1}{\hat{z} - \bar{z}_k} \right] = 0$$

and

$$(3.1) \quad 0 = \Im \left(\frac{p'(\hat{z})}{p(\hat{z})} \right) = \hat{y} \sum_{j=1}^m \frac{1}{|\hat{z} - x_j|^2} + \sum_{k=1}^d \left[\frac{\hat{y} - \beta_k}{|\hat{z} - z_k|^2} + \frac{\hat{y} + \beta_k}{|\hat{z} - \bar{z}_k|^2} \right].$$

Simplifying the right-hand side of (3.1) yields,

$$\sum_{j=1}^m \frac{1}{|\hat{z} - x_j|^2} + 2 \sum_{k=1}^d \frac{(\hat{x} - \alpha_k)^2 + \hat{y}^2 - \beta_k^2}{|\hat{z} - z_k|^2 |\hat{z} - \bar{z}_k|^2} = 0.$$

Thus the numbers $(\hat{x} - \alpha_k)^2 + \hat{y}^2 - \beta_k^2$ cannot all be positive. Hence, there is a positive integer k such that $(\hat{x} - \alpha_k)^2 + \hat{y}^2 \leq \beta_k^2$; that is, \hat{z} lies in or on a Jensen circle of $p(z)$.

Any multiple zeros of $p(z)$ must lie on the boundary of the Jensen circle of p since they are zeros of both p and p' . \square

Jensen's Theorem can be extended to the following theorem.

Theorem 3.5. (Jensen-Nagy-Walsh Theorem [4]). *Let $\varphi(x) \in \mathcal{L} - \mathcal{P}^*$. Then every nonreal zero of $\varphi'(x)$ lies in or on a Jensen circle of $\varphi(x)$.*

Proof. Suppose that $\varphi \in \mathcal{L} - \mathcal{P}^*$. Then we can write $\varphi(z) = \psi(z)p(z)$ where $\psi \in \mathcal{L} - \mathcal{P}$ and $p(z)$ is a real polynomial. Let $z_k = \alpha_k + i\beta_k$ be the (nonzero) zeros of $p(z)$, so we can write $p(z) = d \cdot \prod_{k=1}^m (z - z_k)$, where d is a constant. Let a_k be the zeros of $\psi(z)$. Let $\hat{z} = \hat{x} + i\hat{y}$ be a zero of $\varphi'(z)$ that does not lie in or on a Jensen circle of φ .

Since $\psi \in \mathcal{L} - \mathcal{P}$, we can write

$$\psi(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k} \right) e^{-\frac{z}{a_k}},$$

where c, β , and a_k are real, $\alpha \geq 0$, $n \in \mathbb{Z}^+$ and $\sum a_k^{-2} < \infty$. Now, taking the logarithmic derivative,

$$\frac{\psi'(z)}{\psi(z)} + \frac{p'(z)}{p(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_{k=1}^{\infty} \left(\frac{-z}{a_k(a_k + z)} \right) + \sum_{k=1}^m \left(\frac{1}{z - z_k} \right).$$

Hence,

$$\begin{aligned} \Im \left(\frac{n}{\hat{z}} \right) &= \frac{-n\hat{y}}{\hat{x}^2 + \hat{y}^2} < 0, \\ \Im(-2\alpha\hat{z}) &= -2\alpha\hat{y} < 0, \\ \Im(\beta) &= 0, \text{ and} \\ \Im \left(\frac{-\hat{z}}{a_k(a_k + \hat{z})} \right) &= \frac{-\hat{y}a_k^2}{(a_k^2 + a_k\hat{x})^2 + (a_k\hat{y})^2} < 0. \end{aligned}$$

From the proof of Jensen's Theorem we have,

$$\Im \left(\frac{p'(\hat{z})}{p(\hat{z})} \right) = -\hat{y} \sum_{k=1}^m \frac{1}{|\hat{z} - z_k|} - 2\hat{y} \sum_{k=1}^m \frac{\hat{y}^2 - \beta_k^2 + (\hat{x} - \alpha_k)^2}{|\hat{z} - z_k|^2 |\hat{z} - \bar{z}_k|^2}.$$

Thus, we conclude that

$$\operatorname{sgn} \left(\Im \left(\frac{\psi'(\hat{z})}{\psi(\hat{z})} + \frac{p'(\hat{z})}{p(\hat{z})} \right) \right) = -\operatorname{sgn}(\hat{y}).$$

Since \hat{z} was a zero of $\varphi'(z)$ we have,

$$0 = \frac{\varphi'(\hat{z})}{\varphi(\hat{z})} = \frac{\psi'(\hat{z})}{\psi(\hat{z})} + \frac{p'(\hat{z})}{p(\hat{z})}.$$

But, the right side of this equation has imaginary part strictly less than zero, so it must be the case that \hat{z} lies in or on a Jensen circle of φ . \square

In order to motivate Laguerre's Separation Theorem, we associate with a complex polynomial $p(z)$ of degree n a "generalized" derivative $q(z) := np(z) + (\zeta - z)p'(z)$, $\zeta \in \mathbb{C}$, called the *polar derivative with respect to ζ of $p(z)$* . Now by the Gauss-Lucas Theorem (Theorem 3.1), any disk which contains all the zeros of a complex polynomial $p(z)$, also contains all the zeros of $p'(z)$. What is the corresponding result for polar derivatives? By considering circular regions (i.e., closed disks or the closure of the exterior of such disks or closed half planes), which are "invariant" under Möbius transformations, Laguerre obtained the following invariant form of the Gauss-Lucas Theorem.

Theorem 3.6. (Laguerre's Separation Theorem [1, p. 20]). *Suppose that p is a complex polynomial of degree n with all its zeros in a disk D . Let $\zeta \in \mathbb{C}$ and let w be any zero of*

$$q(z) := np(z) + (\zeta - z)p'(z).$$

If $\zeta \notin D$, then w lies in D .

Proof. Suppose p is a complex polynomial with all of its zeros in a disk D . Pick $\zeta \in \mathbb{C}$ with $\zeta \notin D$. Consider the function $r(z) := p(z)(z - \zeta)^{-n}$. Taking the logarithmic derivative we get,

$$\frac{r'(z)}{r(z)} = \frac{p'(z)}{p(z)} - \frac{n}{z - \zeta} = \frac{p'(z)}{p(z)} + \frac{n}{\zeta - z}.$$

It follows that,

$$r'(z) = \frac{p'(z)}{p(z)} \cdot r(z) + \frac{n}{\zeta - z} \cdot r(z) = p'(z)(z - \zeta)^{-n} + n(\zeta - z)^{-1}(z - \zeta)^{-n}p(z)$$

or

$$r'(z)(\zeta - z)(z - \zeta)^n = p'(z)(\zeta - z) + np(z) = q(z).$$

Let w be a zero of q that does not lie in D . Then $r'(w) = 0$. Note that w cannot be ζ since $q(\zeta) = np(\zeta) \neq 0$ because $\zeta \notin D$. Now, observe that $r(z) = s\left(\frac{1}{z - \zeta}\right)$ for some complex polynomial s of degree n . Also, $r'(w) = s'((w - \zeta)^{-1}) = 0$. Since $\zeta \notin D$ we have $\tilde{D} := \left\{\frac{1}{z - \zeta} : z \in D\right\}$ is also a disk. (This is true because $\frac{1}{z - \zeta}$ is a linear fractional transformation with $0 \cdot \zeta - 1 \cdot 1 = -1 \neq 0$, and must map circles to circles.) This implies that all of the zeros of s must lie in \tilde{D} (since all of the zeros of p lie in D). Thus, by the Gauss-Lucas Theorem 3.1 all of the zeros of s' lie in \tilde{D} . But, $w \notin D$ implies that $(w - \zeta)^{-1} \notin \tilde{D}$, and $s'((w - \zeta)^{-1}) = 0$. A contradiction, so it must be the case that $w \in D$. \square

Remark 3.7. Note that the proof of Laguerre's Separation Theorem does not require that D be an open or closed disk. In fact, D can be a half plane and so the boundary of D can be a straight line.

By way of application of Laguerre's Separation Theorem, we next prove a result on the locus of the critical points of the product and ratio of two complex polynomials.

Theorem 3.8. (Walsh's Two Circle Theorem [1, pp. 20-21]). *Suppose that p is a complex polynomial of degree n and has all n of its zeros in the disk D_1 with center c_1 and radius r_1 . Suppose that q is a complex polynomial of degree m and has all of its m zeros in the disk D_2 with center c_2 and radius r_2 . Then*

- (1) *All the zeros of $(pq)'$ lie in $D_1 \cup D_2 \cup D_3$, where D_3 is the disk with center c_3 and radius r_3 given by*

$$(3.2) \quad c_3 := \frac{nc_2 + mc_1}{n + m}, \quad r_3 := \frac{nr_2 + mr_1}{n + m}.$$

- (2) *All the zeros of $\left(\frac{p}{q}\right)'$ lie in $D_1 \cup D_2 \cup D_3$, where D_3 is the disk with center c_3 and radius r_3 given by*

$$(3.3) \quad c_3 := \frac{nc_2 - mc_1}{n - m}, \quad r_3 := \frac{nr_2 + mr_1}{|n - m|}.$$

Proof. (1) Let z_0 be a zero of $(pq)'$ that lies outside of D_1 and D_2 . Define

$$(3.4) \quad \zeta_1 := z_0 - \frac{np(z_0)}{p'(z_0)} \quad \zeta_2 := z_0 - \frac{mq(z_0)}{q'(z_0)}.$$

Since $z_0 \notin D_1 \cup D_2$, we get $p(z_0) \neq 0 \neq q(z_0)$. By the Gauss-Lucas Theorem (Theorem 3.1) any zero of p' must lie in D_1 and any zero of q' must lie in D_2 . Thus, $p'(z_0) \neq 0 \neq q'(z_0)$.

Now, define $\hat{p}(z) := np(z) + (\zeta_1 - z)p'(z)$. Then

$$\hat{p}(z_0) = np(z_0) + (\zeta_1 - z_0)p'(z_0) = np(z_0) + \left(-\frac{np(z_0)}{p'(z_0)}\right)p'(z_0) = 0.$$

So z_0 is a zero of \hat{p} . By the contrapositive of Laguerre's Separation Theorem, $z_0 \notin D_1 \Rightarrow \zeta_1 \in D_1$. By defining \hat{q} in a similar way, we get $z_0 \notin D_2 \Rightarrow \zeta_2 \in D_2$. Note that

$$\begin{aligned} \frac{n\zeta_2 + m\zeta_1}{n+m} &= \frac{n(z_0 - \frac{mq(z_0)}{q'(z_0)}) + m(z_0 - \frac{np(z_0)}{p'(z_0)})}{n+m} \\ &= \frac{nz_0 - \frac{nmq(z_0)}{q'(z_0)} + mz_0 - \frac{nmp(z_0)}{p'(z_0)}}{m+n} \\ &= \frac{(n+m)z_0}{n+m} - \frac{nm}{m+n} \left(\frac{q(z_0)}{q'(z_0)} + \frac{p(z_0)}{p'(z_0)} \right) \\ &= z_0 - \frac{nm}{n+m} \left(\frac{p'(z_0)q(z_0) + q'(z_0)p(z_0)}{p'(z_0)q'(z_0)} \right) = z_0. \end{aligned}$$

Since $z_0 \notin D_1$ we know that $|z_0 - c_1| > r_1$ and since $z_0 \notin D_2$, we know that $|z_0 - c_2| > r_2$. We now show that $|z_0 - c_3| < r_3$, where c_3 and r_3 are defined by (3.2). Consider

$$\begin{aligned} |z_0 - c_3| &= \left| \frac{n\zeta_2 + m\zeta_1}{n+m} - \frac{nc_2 + mc_1}{n+m} \right| \\ &= \left| \frac{n(\zeta_2 - c_2) + m(\zeta_1 - c_1)}{n+m} \right| \\ &\leq \frac{n|\zeta_2 - c_2| + m|\zeta_1 - c_1|}{n+m} \\ &\leq \frac{nr_2 + mr_1}{n+m} = r_3. \end{aligned}$$

Therefore, $z_0 \in D_3$.

(2) The proof for part 2 is similar to part 1. Let z_0 be a zero of $(\frac{p}{q})'$ and let ζ_1 and ζ_2 be defined by (3.4). Then note that $\frac{n\zeta_2 - m\zeta_1}{n-m} = z_0$ (using the fact that z_0 was a zero of the derivative of $\frac{p}{q}$). It can then be shown that $|z_0 - c_3| \leq r_3$, where c_3 and r_3 are defined by (3.3). Hence, $z_0 \in D_3$. \square

4. NECESSARY AND SUFFICIENT CONDITIONS

We have seen in Section 2 that the Laguerre inequality is a necessary condition for a real entire function φ to belong to the Laguerre-Pólya class $(\mathcal{L} - \mathcal{P})$. Here we formulate several necessary *and* sufficient conditions for φ to be in $\mathcal{L} - \mathcal{P}$. In particular, we state the extended Laguerre inequality (Theorem 4.1) and the complex versions of the Laguerre inequality (Theorem 4.2 and Theorem 4.3). Under additional assumptions another characterization of functions in $\mathcal{L} - \mathcal{P}$ is also presented (Theorem 4.5). The characterization of polynomials in $\mathcal{L} - \mathcal{P}$ in terms of the metric properties of the logarithmic derivative appears to be new (Theorem 4.8). Our final result (Theorem 4.9) of this section may be of interest since it provides a necessary and sufficient condition for the validity of the Riemann Hypothesis.

The following theorem refers to what was previously termed the extended Laguerre inequalities.

Theorem 4.1. ([7, Theorem 2.9]). *Let*

$$f(z) := e^{-\alpha z^2} f_1(z) \quad (\alpha \geq 0, f(z) \not\equiv 0),$$

where $f_1(z)$ is a real entire function of genus 0 or 1. Set

$$L_n[f](x) := \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x) \quad \forall x \in \mathbb{R}; \quad n = 0, 1, 2, \dots$$

Then, $f(z) \in \mathcal{L} - \mathcal{P}$ if and only if

$$L_n[f](x) \geq 0 \quad \text{for all } x \in \mathbb{R}; \quad n = 0, 1, 2, \dots$$

The next two theorems characterize functions in $\mathcal{L} - \mathcal{P}$ in terms of the complex analogs of the Laguerre inequality.

Theorem 4.2. ([7, Theorem 2.10]). *Let $f(z)$ be the real entire function as defined in Theorem 4.1. Then, $f(z) \in \mathcal{L} - \mathcal{P}$ if and only if*

$$(4.1) \quad |f'(z)|^2 \geq \Re\{f(z)\overline{f''(z)}\} \quad \text{for all } z \in \mathbb{C}.$$

The proof of Theorem 4.2 is based on the geometric interpretation of the inequality (4.1). Indeed, if $f \in \mathcal{L} - \mathcal{P}$, then a computation shows that

$$(4.2) \quad \frac{\partial^2}{\partial y} |f(x + iy)|^2 = 2|f'(z)|^2 - 2\Re\{f(z)\overline{f''(z)}\}.$$

Thus, (4.2) says that $|f(x + iy)|^2$ is a convex function of y .

We have included here the proof of the following theorem to provide a comparison with the proof of Theorem 6.22.

Theorem 4.3. ([7, Theorem 2.12]). *Let $f(z)$ be a real entire function of the form*

$$f(z) := Ce^{-\alpha z^2 + \beta z} z^n \prod_{k=1}^{\omega} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k}} \quad (\omega \leq \infty),$$

where $\alpha \geq 0$, C , and β are real numbers, n is a nonnegative integer, and the z_k 's are nonzero with $\sum_{k=1}^{\omega} |z_k|^{-2} < \infty$, and the zeros $\{z_k\}_{k=1}^{\infty}$ of $f(z)$ are counted according to multiplicity and are arranged so that $0 < |z_1| \leq |z_2| \leq \dots$. Then, $f(z) \in \mathcal{L} - \mathcal{P}$ if and only if

$$(4.3) \quad \frac{1}{y} \Im\{-f'(z)\overline{f(z)}\} \geq 0 \quad \text{for all } z = x + iy \in \mathbb{C}, \quad y \neq 0.$$

Proof. Let $z = x + iy$ and $z_k = x_k + iy_k$. By calculation we have,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= -2\alpha z + \beta + \frac{n}{z} + \sum_{k=1}^{\omega} \left(\frac{-\frac{1}{z_k}}{1 - \frac{z}{z_k}} + \frac{1}{z_k} \right) \\ &= -2\alpha z + \beta + \frac{n}{z} + \sum_{k=1}^{\omega} \left(\frac{-1}{z_k - z} + \frac{1}{z_k} \right) \\ &= -2\alpha(x + iy) + \beta + \frac{n(x - iy)}{x^2 + y^2} \\ &\quad + \sum_{k=1}^{\omega} \left(\frac{-(x_k - x) + i(y_k - y)}{(x_k - x)^2 + (y_k - y)^2} + \frac{x_k - iy_k}{x_k^2 + y_k^2} \right). \end{aligned}$$

Thus,

$$\frac{1}{y} \Im\left(-\frac{f'(z)}{f(z)}\right) = 2\alpha + \frac{n}{x^2 + y^2} + \sum_{k=1}^{\omega} \left(\frac{1 - \frac{y_k}{y}}{(x_k - x)^2 + (y_k - y)^2} + \frac{\frac{y_k}{y}}{x_k^2 + y_k^2} \right)$$

and

$$R(z) := \frac{1}{y} \Im \left(-\frac{f'(z)}{f(z)} \right) = \frac{1}{y|f(z)|^2} \Im(-f'(z)\overline{f(z)}).$$

If $f(z) \in \mathcal{L} - \mathcal{P}$, then $y_k = 0$ for all k . Thus, $R(z) \geq 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Conversely, suppose that (4.3) holds. We will show that the assumption that $f(z)$ has a nonreal zero leads to a contradiction. Without loss of generality we may assume that $y_1 \neq 0$. Pick ϵ such that $1 > \epsilon > 0$ and consider the following,

$$\begin{aligned} R(x_1 + iy_1(1 - \epsilon)) &= 2\alpha + \frac{n}{x_1^2 + y_1^2(1 - \epsilon)^2} + \\ &\quad \left(\frac{1 - \frac{y_1}{y_1(1 - \epsilon)}}{(x_1 - x_1)^2 + (y_1 - y_1(1 - \epsilon))^2} + \frac{\frac{y_1}{y_1(1 - \epsilon)}}{x_1^2 + y_1^2} \right) \\ &\quad + S_2(x_1 + iy_1(1 - \epsilon)) \\ &= 2\alpha + \frac{n}{x_1^2 + y_1^2(1 - \epsilon)^2} - \frac{1}{\epsilon y_1^2(1 - \epsilon)} + \frac{1}{(1 - \epsilon)(x_1^2 + y_1^2)} \\ &\quad + S_2(x_1 + iy_1(1 - \epsilon)), \end{aligned}$$

where

$$S_2(x_1 + iy_1(1 - \epsilon)) := \sum_{k=2}^{\omega} \left[\frac{1 - \frac{y_k}{y_1(1 - \epsilon)}}{(x_1 - x_k)^2 + [y_1(1 - \epsilon) - y_k]^2} + \frac{\frac{y_k}{y_1(1 - \epsilon)}}{x_k^2 + y_k^2} \right].$$

As $\epsilon \rightarrow 0^+$, $R(x_1 + iy_1(1 - \epsilon)) \rightarrow -\infty$, because $\frac{1}{\epsilon y_1^2(1 - \epsilon)} \rightarrow \infty$ as $\epsilon \rightarrow 0$. This contradicts (4.3) and thus the proof is complete. \square

Simple examples show that the Laguerre inequality $L[f](x) \geq 0 \ \forall x \in \mathbb{R}$ is *only* a *necessary condition* for a real entire function, f , to belong to $\mathcal{L} - \mathcal{P}$. Indeed, if $f(x) = x^4 - 1$, then $L[f](x) = 4x^2(3 + x^4) \geq 0$ for all $x \in \mathbb{R}$. In [6] the authors' main goal is to investigate conditions on the Laguerre expression, $L[g] = (g')^2 - gg''$, of a function $g \in \mathfrak{S}(A)$ that imply that g has only real zeros; that is, $g \in \mathcal{L} - \mathcal{P}$. Consider, for example, $g(x) = e^x x(x^2 + 1) \in \mathfrak{S}(1)$. Then $L[g](x) = e^{2x}(1 + 3x^4) > 0$ for all $x \in \mathbb{R}$ and thus we see that some hypotheses beyond the *strict* inequality $L[g](x) > 0 \ (\forall x \in \mathbb{R})$ are required in order that we can conclude that g has only real zeros. The additional hypotheses used in [6] involve the family of real entire functions

$$g_\lambda(x) = g(x + i\lambda) + g(x - i\lambda) \quad (\lambda \in \mathbb{R})$$

closely related to the real entire function g . We remark that if $g \in \mathfrak{S}(A)$, then $g_\lambda(z) = 2 \cos(\lambda D)g(z)$, where D denotes differentiation with respect to z . In [6] the authors investigate the level set structure of a real entire function, f , to establish a connection between the level sets of f , the Laguerre expression for f and the distribution of zeros of f . Since these deeper studies are outside the scope of the present paper, here we merely cite, by way of illustration, two recent results dealing with the Laguerre inequality.

Theorem 4.4. ([6, Theorem I]). *If $f \in \mathcal{L} - \mathcal{P}$ and if f is not of the form Ce^{bz} , then $L[f_\lambda](x) > 0$ for all $\lambda \neq 0$ and for all $x \in \mathbb{R}$.*

The following theorem follows from Theorem 4.4.

Theorem 4.5. ([6, Corollary I]). *Suppose that $f \in \mathcal{L} - \mathcal{P}^*$ and f is not of the form Ce^{bz} . Then $f \in \mathcal{L} - \mathcal{P}$ if and only if $L[f_\lambda](x) > 0$ for all $\lambda \neq 0$ and for all $x \in \mathbb{R}$.*

We remark that it is not known if the converse of Theorem 4.4 is valid in the absence of some additional assumptions.

The metric properties of the logarithmic derivative of a polynomial are not widely known and for this reason we have included here the proof of the following theorem.

Theorem 4.6. ([1, p. 345]). *If p is a real polynomial of degree n and if $p \in \mathcal{L} - \mathcal{P}$, then*

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda}, \quad \forall \lambda > 0,$$

where μ denotes Lebesgue measure.

Proof. Suppose that $\lambda > 0$. First suppose that $p(x)$ has distinct zeros, $\alpha_1 < \alpha_2 < \dots < \alpha_n$. Then

$$\frac{p'(x)}{\lambda p(x)} = \frac{1}{\lambda} \sum_{k=1}^n \frac{1}{x - \alpha_k}.$$

Let $\beta_1 < \beta_2 < \dots < \beta_n$ be the zeros of $\lambda p - p'$. Note that at each β_k , $\frac{p'(\beta_k)}{p(\beta_k)} = \lambda$.

An examination of the graph of $\frac{p'(x)}{\lambda p(x)}$ shows that,

$$\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\} = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup \dots \cup [\alpha_n, \beta_n].$$

Thus,

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \sum_{k=1}^n (\beta_k - \alpha_k) = \sum_{k=1}^n \beta_k - \sum_{k=1}^n \alpha_k.$$

If $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$, then $\sum_{k=1}^n \alpha_k = -a_{n-1}$. Note that the zeros of $\lambda p - p'$ are the same as the zeros of $p - \lambda^{-1}p'$. Thus, $\sum_{k=1}^n \beta_k = -(a_{n-1} - \frac{n}{\lambda})$. Therefore,

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \sum_{k=1}^n \beta_k - \sum_{k=1}^n \alpha_k = -a_{n-1} + \frac{n}{\lambda} - (-a_{n-1}) = \frac{n}{\lambda}.$$

□

Remark 4.7. We remark that Theorem 4.6 does not hold if $f(x) \notin \mathcal{L} - \mathcal{P}$. Consider for example, $f(x) = x^2 + 1$ with $\lambda = 2$.

The converse of Theorem 4.6 is also true, and appears to be new.

Theorem 4.8. *If p is a real polynomial of degree n with,*

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda}, \quad \forall \lambda > 0,$$

then $p \in \mathcal{L} - \mathcal{P}$.

Proof. Suppose that

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda}, \quad \forall \lambda > 0.$$

We first show that

$$\lim_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = Z_R(p),$$

where $Z_R(p)$ denotes the number of real zeros of p , counting multiplicities.

Let $p(x) = f(x)g(x)$ where $g(x) \in \mathcal{L} - \mathcal{P}$ and $f(x)$ has only nonreal zeros. If $p(x) \in \mathcal{L} - \mathcal{P}$, then set $f(x) = 1$ and if $p(x)$ has no real zeros, then set $g(x) = 1$. Note that in the case when $p \in \mathcal{L} - \mathcal{P}$ we are done. Consider now

$$\frac{p'(x)}{p(x)} = \frac{g'(x)}{g(x)} + \frac{f'(x)}{f(x)}.$$

Since $f(x)$ has no real zeros, $\frac{f'}{f}$ has no vertical asymptotes and since the degree of the numerator is less than the degree of the denominator the function $\frac{f'(x)}{f(x)}$ must be bounded. Thus, there exists $m, M > 0$ such that

$$-m \leq \frac{f'(x)}{f(x)} \leq M, \quad \forall x \in \mathbb{R}.$$

Let $\lambda > \max\{m, M\} := \hat{M}$. Then for $\lambda > \hat{M}$ we have,

$$\begin{aligned} x \in \left\{ x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda \right\} &\Rightarrow \frac{p'(x)}{p(x)} = \frac{g'(x)}{g(x)} + \frac{f'(x)}{f(x)} \geq \lambda \\ &\Rightarrow \frac{g'(x)}{g(x)} \geq \lambda - \frac{f'(x)}{f(x)} \geq \lambda - \hat{M} \\ &\Rightarrow x \in \left\{ x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda - \hat{M} \right\}. \end{aligned}$$

Hence,

$$\left\{ x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda \right\} \subset \left\{ x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda - \hat{M} \right\}$$

which implies that

$$\begin{aligned} \mu\left(\left\{ x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda \right\}\right) &\leq \mu\left(\left\{ x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda - \hat{M} \right\}\right) \\ &= \frac{Z_R(g)}{\lambda - \hat{M}} \quad (\text{by Theorem 4.6}) \\ &= \frac{Z_R(p)}{\lambda - \hat{M}}. \end{aligned}$$

We now have

$$\limsup_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{ x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda \right\}\right) \leq \limsup_{\lambda \rightarrow \infty} \frac{\lambda Z_R(p)}{\lambda - \hat{M}} = Z_R(p).$$

Let

$$x \in \left\{ x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda + \hat{M} \right\}.$$

There are now two cases to consider:

(1) If $\frac{f'(x)}{f(x)} \geq 0$, then

$$\frac{g'(x)}{g(x)} \geq \lambda + \hat{M} \geq \lambda + \frac{f'(x)}{f(x)} \geq \lambda - \frac{f'(x)}{f(x)}.$$

(2) If $\frac{f'(x)}{f(x)} < 0$, then

$$0 < -\frac{f'(x)}{f(x)} < \hat{M}.$$

Consequently

$$\frac{g'(x)}{g(x)} \geq \lambda + \hat{M} \geq \lambda - \frac{f'(x)}{f(x)}.$$

Thus,

$$\left\{x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda + \hat{M}\right\} \subset \left\{x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda - \frac{f'(x)}{f(x)}\right\} = \left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}$$

and

$$\frac{Z_R(p)}{\lambda + \hat{M}} = \mu\left(\left\{x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda + \hat{M}\right\}\right) \leq \mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right).$$

Hence,

$$\begin{aligned} \liminf_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) &\geq \liminf_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{x \in \mathbb{R} : \frac{g'(x)}{g(x)} \geq \lambda + \hat{M}\right\}\right) \\ &= \liminf_{\lambda \rightarrow \infty} \frac{\lambda Z_R(p)}{\lambda + \hat{M}} = Z_R(p). \end{aligned}$$

By the above lim sup and lim inf arguments, we conclude that

$$\lim_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = Z_R(p).$$

Since by assumption

$$\mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda}, \quad \forall \lambda \in \mathbb{R},$$

therefore,

$$\lim_{\lambda \rightarrow \infty} \lambda \mu\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \lim_{\lambda \rightarrow \infty} \frac{\lambda n}{\lambda} = n = Z_R(p).$$

□

In order to state the final result of this section, it will be convenient to adopt here the following terminology. A function (kernel) $K : \mathbb{R} \rightarrow \mathbb{R}^+$ is called an *admissible* kernel, if (i) $K(t)$ is analytic in the strip $S(A)$ for some $A > 0$, (ii) K is even, (iii) $K'(t) < 0$ for $t > 0$ and (iv) for some $\epsilon > 0$ and $n = 0, 1, 2, \dots$,

$$K^{(n)}(t) = O(\exp(-|t|^{2+\epsilon})) \quad \text{as } t \rightarrow \infty.$$

Theorem 4.9. ([7, Theorem 2.4]). *Let $K(t)$ be an admissible kernel and let*

$$F(x) = \int_0^\infty K(t) \cos xt \, dt.$$

Set

$$c_{m,n}(\alpha) := \int_{-\infty}^\infty (it)^m (\alpha + it)^n \Phi(t) \, dt \quad (\alpha \in \mathbb{R}; m, n = 0, 1, 2, \dots).$$

Then, $F(x) \in \mathcal{L} - \mathcal{P}$ if and only if the moments $c_{m,n}(\alpha)$ satisfy the Turán inequalities

$$(4.4) \quad I_n(\alpha) := c_{1,n-1}^2(\alpha) - c_{0,n-1}(\alpha)c_{2,n-1}(\alpha) > 0 \quad (\alpha \in \mathbb{R}; n = 1, 2, 3, \dots).$$

Now it is known that the *Jacobi theta function*, $\Phi(t)$, where

$$\Phi(t) = \sum_{n=1}^{\infty} \pi n^2 (2\pi n^2 e^{4t} - 3) \exp(5t - \pi n^2 e^{4t}) \quad x \in \mathbb{R},$$

is an admissible kernel ([7, Theorem A]) and it is known that the Riemann Hypothesis is equivalent to the statement that

$$F(x) = \int_0^{\infty} \Phi(t) \cos xt \, dt \in \mathcal{L} - \mathcal{P}.$$

Thus, by Theorem 4.9, the Riemann Hypothesis is valid if and only if $I_n(\alpha) > 0$ for all $\alpha \in \mathbb{R}$ and $n = 0, 1, 2, \dots$, where $I_n(\alpha)$ is defined by (4.4).

5. THE CENTER OF MASS WITH RESPECT TO A FINITE POINT

For a set of n complex numbers z_1, z_2, \dots, z_n we define the *center of mass* of these numbers to be $\zeta = \frac{1}{n} \sum_{k=1}^n z_k$. It is worth noting that a translation or rotation of each of the z_k will result in a translation or rotation, respectively, of ζ ; i.e., if $z_k \mapsto z_k + a$ for all z_k , then $\zeta \mapsto \zeta + a$ and if $z_k \mapsto e^{i\theta} z_k$ for all z_k , then $\zeta \mapsto e^{i\theta} \zeta$. Also from the definition, if $\Re a \leq \zeta \leq \Re b$, then $\Re a \leq \zeta \leq \Re b$. The above statements lead to the following proposition.

Proposition 5.1. *If C is the smallest convex polygon that contains the complex numbers z_k ($1 \leq k \leq n$), then the center of mass, ζ , also belongs to that polygon.*

Proof. First, rotate the polygon so that one of the sides is vertical and the rest of the polygon is located to the right of that side. If the side is $[z_1, z_2]$, then $\Re(z_1) \leq \Re(z_k)$ for all k . Also, by the above inequality $\Re(z_1) \leq \Re(\zeta)$. The side $[z_1, z_2]$ determines two half planes, one side that contains the z_k and one side that does not. By the previous inequality, ζ lies in the half plane that contains the z_k . Since the convex polygon C is the intersection of these half planes, ζ must lie in C . \square

It is possible to define a generalized center of mass with respect to a specified point. We begin by defining the above center of mass as the center of mass with respect to infinity, ζ_{∞} ; that is,

$$\zeta_{\infty} = \frac{1}{n} \sum_{k=1}^n z_k.$$

To define the center of mass with respect to a point, say z_0 , we first map z_0 to the point at infinity by the linear fractional transformation,

$$z \mapsto \frac{a}{z - z_0} + b.$$

Under this mapping z_k maps to z'_k and z_0 maps to ∞ , where

$$(5.1) \quad z'_k = \frac{a}{z_k - z_0} + b.$$

We now use the previous definition of center of mass and set

$$\zeta'_{\infty} = \frac{1}{n} \sum_{k=1}^n z'_k.$$

Using the transformation (5.1),

$$\zeta'_\infty = \frac{1}{n} \sum_{k=1}^n z'_k = \frac{1}{n} \sum_{k=1}^n \left(\frac{a}{z_k - z_0} + b \right) = b + \frac{a}{n} \sum_{k=1}^n \frac{1}{z_k - z_0}.$$

But we also have that ζ'_∞ is the image of ζ_{z_0} under the transformation (5.1), thus

$$\zeta'_\infty = \frac{a}{\zeta_{z_0} - z_0} + b.$$

Equating the previous two equations we have,

$$\frac{a}{\zeta_{z_0} - z_0} + b = b + \frac{a}{n} \sum_{k=1}^n \frac{1}{z_k - z_0}.$$

Finally, solving for ζ_{z_0} we obtain the formula,

$$(5.2) \quad \zeta_{z_0} = z_0 + n \left(\sum_{k=1}^n \frac{1}{z_k - z_0} \right)^{-1},$$

and we will call (5.2) the (generalized) center of mass of the points z_1, z_2, \dots, z_n with respect to the point z_0 .

The following theorem is a necessary and sufficient condition for a polynomial to have only real zeros. It requires the use of Laguerre's Separation Theorem (cf. Theorem 3.6) which can be restated in terms of the generalized center of mass as follows.

Theorem 5.2. (Laguerre's Separation Theorem (Center of Mass Version)). *Suppose that f is a complex polynomial of degree n with all its zeros in a disk D . Let $\zeta \in \mathbb{C}$ and let a be any zero of*

$$nf(z) + (\zeta - z)f'(z).$$

If $\zeta \notin D$, then $a \in D$, i.e., if ζ_a is the center of mass of the zeros of f with respect to a , then ζ_a and a are on opposite sides of the boundary of D .

Theorem 5.3. ([10, Theorem 6]). *Let ζ_z be the center of mass of the zeros of the polynomial $f(z)$ with respect to z . Then all the zeros of $f(z)$ are real if and only if,*

$$\Im(z)\Im(\zeta_z) = \Im(z)\Im\left(z - n \frac{f(z)}{f'(z)}\right) < 0 \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. Suppose that $f(z) = c \cdot \prod_{k=1}^n (z - z_k)$ and $z_k \in \mathbb{R}$ for all k . Pick $\epsilon > 0$. Let $a \in \mathbb{C}$ be such that $\Im(a) > \epsilon > 0$. Suppose the zeros of f are real and contained in the disk D (note D can be a half plane). Let the circle in the complex plane that corresponds to the boundary of D be the line $y = \epsilon$ (call the circle γ). Suppose that $\zeta_a \in D$, i.e., $\Im(\zeta_a) < \epsilon$. Since ϵ is arbitrary, $\Im(\zeta_a) \leq 0$. It will now be shown that $\Im(\zeta_a) = 0$ will lead to a contradiction of Laguerre's Separation Theorem. If $\Im(\zeta_a) = 0$, then the circle γ can be deformed to a circle γ' that contains the zeros of f but does not contain the points a and ζ_a , contradicting Laguerre's Separation Theorem. Therefore, $\Im(a) \cdot \Im(\zeta_a) < 0$.

Conversely, suppose that $\Im(a) \cdot \Im(\zeta_a) < 0$ and that a zero, say z_1 of f is such that $\Im z_1 \neq 0$. As a approaches z_1 ,

$$\zeta_a = a - n \frac{f(a)}{f'(a)} \longrightarrow z_1.$$

Hence for a sufficiently close to z_1 , $\Im(\zeta_a)$ and $\Im(a)$ have the same sign. This contradicts our assumption that $\Im(a) \cdot \Im(\zeta_a) < 0$ and therefore f has no nonreal zeros. \square

Using the Theorem 5.3 we next give a new proof of a special case of Theorem 4.3. In the sequel it will be convenient to use the notation

$$H^+ := \{z \in \mathbb{C} : \Im(z) > 0\}.$$

Theorem 5.4. (Theorem 4.3). *Let $f(z)$ be a real entire function of the form*

$$f(z) := Ce^{-\alpha z^2 + \beta z} z^n \prod_{k=1}^{\omega} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k}} \quad (\omega \leq \infty),$$

where $\alpha \geq 0$, C , and β are real numbers, n is a nonnegative integer, and the z_k 's are nonzero with $\sum_{k=1}^{\omega} |z_k|^{-2} < \infty$, and the zeros $\{z_k\}_{k=1}^{\infty}$ of $f(z)$ are counted according to multiplicity and are arranged so that $0 < |z_1| \leq |z_2| \leq \dots$. Then, $f(z) \in \mathcal{L} - \mathcal{P}$ if and only if

$$\frac{1}{y} \Im\{-f'(z)\overline{f(z)}\} \geq 0 \quad \text{for all } z = x + iy \in H^+.$$

Theorem 5.5. (Special Case of Theorem 4.3). *A real polynomial $p(z) = \prod_{k=1}^n (z - z_k)$ has only real zeros if and only if it satisfies the property*

$$\Im\left(\frac{p'(z)}{p(z)}\right) < 0, \quad \text{for all } z \in H^+.$$

Proof. Suppose $p(z)$ has only real zeros. By Theorem 5.3, $\Im z \cdot \Im \zeta_z < 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. If $z \in H^+$, then

$$\begin{aligned} \Im z \cdot \Im \zeta_z &= \Im z \left[\Im \left(z - n \left[\sum_{k=1}^n \frac{1}{z - z_k} \right]^{-1} \right) \right] \\ &= \Im z \left[\Im z - n \Im \frac{p(z)}{p'(z)} \right] \\ &= (\Im z)^2 - n \Im z \cdot \Im \left(\frac{p(z)}{p'(z)} \right) < 0. \end{aligned}$$

It follows that

$$0 < (\Im z)^2 < n \Im z \cdot \Im \frac{p(z)}{p'(z)}.$$

Therefore, $\Im \frac{p'(z)}{p(z)} < 0$ for all $z \in H^+$.

Conversely, suppose that $p(z)$ satisfies,

$$\Im\left(\frac{p'(z)}{p(z)}\right) < 0, \quad \text{for all } z \in H^+.$$

Let $z = x + iy$ and $z_k = x_k + iy_k$ for $k = 1, 2, \dots, n$. Then,

$$\begin{aligned} \Im \frac{p'(z)}{p(z)} &= \Im \left(\sum_{k=1}^n \frac{1}{z - z_k} \right) \\ &= \sum_{k=1}^n \frac{-(y - y_k)}{(x - x_k)^2 + (y - y_k)^2} < 0, \quad \forall y > 0. \end{aligned}$$

Let

$$R(z) = R(x + iy) := \frac{-1}{y} \sum_{k=1}^n \frac{1 - \frac{y_k}{y}}{(x - x_k)^2 + (y - y_k)^2}.$$

It will be shown that if one of the $y_k \neq 0$ (i.e. there is a nonreal zero), then there exists a $y > 0$ such that $R(x + iy) > 0$. Without loss of generality suppose that $y_1 > 0$. Let $y_2 = -y_1$ correspond to the conjugate zero of $z_1 = x_1 + iy_1$. Pick $\epsilon > 0$ such that $0 < \epsilon < 1$. Consider R evaluated at the point $x_1 + iy_1(1 - \epsilon)$,

$$\begin{aligned} R(x_1 + iy_1(1 - \epsilon)) &= \frac{-1}{y_1(1 - \epsilon)} \sum_{k=1}^n \frac{1 - \frac{y_k}{y_1(1 - \epsilon)}}{(x_1 - x_k)^2 + (y_1(1 - \epsilon) - y_k)^2} \\ &= \frac{-1}{y_1(1 - \epsilon)} \left[\frac{1 - \frac{y_1}{y_1(1 - \epsilon)}}{(x_1 - x_1)^2 + (y_1(1 - \epsilon) - y_1)^2} + \frac{1 - \frac{y_2}{y_1(1 - \epsilon)}}{(x_1 - x_2)^2 + (y_1(1 - \epsilon) - y_2)^2} \right. \\ &\quad \left. + \sum_{k=3}^n \frac{1 - \frac{y_k}{y_1(1 - \epsilon)}}{(x_1 - x_k)^2 + (y_1(1 - \epsilon) - y_k)^2} \right]. \end{aligned}$$

Thus we have,

$$\begin{aligned} R(x_1 + iy_1(1 - \epsilon)) &= \frac{-1}{y_1(1 - \epsilon)} \left[\frac{-1}{\epsilon(1 - \epsilon)y_1^2} \right. \\ &\quad \left. + \frac{-\epsilon}{(1 - \epsilon)[(x_1 - x_2)^2 + (2y_1 - \epsilon y_1)^2]} + \sum_{k=3}^n \frac{1}{(x_1 - x_k)^2 + (y_1(1 - \epsilon))^2} \right]. \end{aligned}$$

Since $\frac{-1}{\epsilon(1 - \epsilon)y_1^2} \rightarrow -\infty$ as $\epsilon \rightarrow 0^+$, we can conclude that for sufficiently small $\epsilon > 0$, $R(x_1 + iy_1(1 - \epsilon)) > 0$. Therefore, y_1 must be 0 and hence $p(z)$ has only real zeros. \square

6. THE POLAR DERIVATIVE

The polar derivative was first introduced in connection with Laguerre's Separation Theorem (cf. Theorem 3.6). This section covers several properties of the polar derivative that are analogous to the usual derivative. It also contains new theorems and questions related to the polar derivative and what will be termed as the *polar derivative Laguerre expression*.

Let $\zeta \in \mathbb{C}$. Then the polar derivative (nonlinear) operator

$$T_\zeta : \mathbb{C}[z] \rightarrow \mathbb{C}[z]$$

is defined as follows. If $f(z)$ is a complex polynomial of degree n , then

$$T_\zeta[f(z)] := nf(z) + (\zeta - z)f'(z).$$

Lemma 6.1 (Binomial Form of the Polar Derivative). *Let $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$. Then for a fixed $\zeta \in \mathbb{C}$,*

$$T_\zeta[f](z) = n \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + \zeta a_{k+1}) z^k.$$

Proof. By a calculation we have,

$$\begin{aligned}
T_\zeta[f](z) &= n \sum_{k=0}^n \binom{n}{k} a_k z^k + (\zeta - z) \sum_{k=1}^n \binom{n}{k} a_k k z^{k-1} \\
&= \sum_{k=1}^{n-1} n \binom{n}{k} a_k z^k + \sum_{k=1}^{n-1} \zeta \binom{n}{k} a_k k z^{k-1} - \sum_{k=1}^{n-1} \binom{n}{k} a_k k z^k \\
&\quad + n a_n z^n + \zeta a_n n z^{n-1} - a_n n z^n + n a_0 \\
&= \sum_{k=1}^{n-1} \binom{n}{k} (n a_k z^k - a_k k z^k) + \sum_{k=1}^{n-1} \zeta \binom{n}{k} a_k k z^{k-1} + \zeta a_n n z^{n-1} + n a_0 \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (n - k) a_k z^k + \sum_{k=0}^{n-1} \zeta \binom{n}{k+1} (k+1) a_{k+1} z^k \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + \zeta a_{k+1}) z^k.
\end{aligned}$$

□

Remark 6.2. We remark here that any polynomial can be written in the form $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ by simply factoring out $\binom{n}{k}$ from the k^{th} coefficient of f . It is because of Lemma 6.1 and the following lemma that it is advantageous to work with polynomials expressed in this particular form.

Lemma 6.3. *Let $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$. Then the degree of $T_\zeta[f](z)$ is $n - 1$ for all $\zeta \neq -\frac{a_{n-1}}{a_n}$.*

Proof. By Lemma 6.1 if $\zeta = -\frac{a_{n-1}}{a_n}$ then the z^{n-1} disappears and the degree of $T_\zeta[f]$ cannot be $n - 1$. □

Remark 6.4. We remark here that in [15, p. 57] there appears to be a misprint regarding the degree of the polar derivative of a polynomial of degree n .

Remark 6.5. Lemma 6.3 says that computation of the second polar derivative of a polynomial requires that $\zeta \neq -\frac{a_{n-1}}{a_n}$. Since $T_\zeta[T_\zeta[f]](x) = (n - 1)T_\zeta[f](x) + (\zeta - x)T_\zeta[f]'(x)$ and $\deg(T_\zeta[f](x)) = n - 1$.

Remark 6.6. We note that $T_\zeta[f](x) \equiv 0$ if and only if $f(z) = (z - \zeta)^n$. Suppose that $f(z) = (z - \zeta)^n$. Then,

$$T_\zeta[f](z) = n(z - \zeta)^n + (\zeta - z)[n(z - \zeta)^{n-1}] = n(z - \zeta)^n - n(z - \zeta)^n = 0.$$

In this case, the polar derivative is a constant with degree 0.

Now, suppose that the polar derivative of $f(z)$ is identically 0; i.e., $T_\zeta[f](z) \equiv 0$, and $f(z)$ is not identically 0. Then,

$$nf(z) = (z - \zeta)f'(z).$$

It follows that,

$$\frac{f'(z)}{f(z)} = \frac{n}{z - \zeta} = \sum_{k=1}^n \frac{1}{z - \zeta}.$$

Thus, $f(z) = (z - \zeta)^n$.

Henceforth, we will assume, unless stated otherwise, that $\zeta \in \mathbb{R}$.

Theorem 6.7. (Polar Derivative Analog of Rolle's Theorem). *Suppose that $f(x)$ is a real polynomial of degree n . Then between any two consecutive real zeros of $f(x)$, say $x_1 < x_2$, there is at least one real zero of $T_\zeta[f](x)$ in (x_1, x_2) for all $\zeta \notin [x_1, x_2]$.*

Proof. Suppose that $x_1 < x_2$ are two consecutive zeros of $f(x)$. Since

$$T_\zeta[f](x_1)T_\zeta[f](x_2) = (\zeta - x_1)(\zeta - x_2)f'(x_1)f'(x_2) < 0,$$

there must be a sign change of $T_\zeta[f](x)$ on the interval (x_1, x_2) . Therefore, $T_\zeta[f](x)$ has a zero in (x_1, x_2) . \square

Remark 6.8. We remark here that, in general, it is necessary for $\zeta \notin [x_1, x_2]$. Consider the following example. If $f(x) = x^2 - 4$, then $T_0[f](x) = -8$ which has no zeros. We can also consider $T_1[f](x) = 2(x - 4)$ which has a real zero but, not in the interval $(-2, 2)$.

Theorem 6.9. *If $f(x)$ is a polynomial of degree n , then for any $\zeta \in \mathbb{R}$*

$$(6.1) \quad Z_C(T_\zeta[f](x)) \leq Z_C(f(x))$$

where $Z_C(f)$ denotes the number of nonreal zeros of a polynomial, counting multiplicities.

Proof. Suppose that there are $2d$ nonreal zeros of $f(x)$ and m real simple zeros of $f(x)$. Then, by the Polar Derivative Analog of Rolle's Theorem (Theorem 6.7) we know that $T_\zeta[f](x)$ has at least one real zero in each of the $m - 2$ intervals determined by the real zeros of f (since ζ can be in at most one of the $m - 1$ intervals). Then by degree considerations we conclude that

$$\begin{aligned} Z_C(T_\zeta[f]) &= \deg T_\zeta[f] - Z_R(T_\zeta[f]) \\ &\leq 2d + m - 1 - m + 2 = 2d + 1. \end{aligned}$$

Since $Z_C(T_\zeta[f])$ must be even,

$$Z_C(T_\zeta[f]) \leq 2d = Z_C(f).$$

Suppose that f has multiple zeros. Each multiple zero of f is a zero of $T_\zeta[f]$. Hence, there will still be at least $m - 2$ real zeros of $T_\zeta[f]$ and the proof will be the same as above. \square

Remark 6.10. The following alternate proof of Theorem 6.9 is of interest since it shows that inequality (6.1) follows from the special case when $\zeta = 0$. Let $p(x) = \sum_{k=0}^n a_k x^k$ be a real polynomial of degree n . Let

$$h(x) = T_0[p](x) = np(x) - xp'(x).$$

Then

$$(6.2) \quad Z_C(h) \leq Z_C(p).$$

To prove (6.2), we first suppose that $a_0 a_{n-1} \neq 0$. Let

$$p^*(x) = x^n p\left(\frac{1}{x}\right) = \sum_{k=0}^n a_k x^{n-k}.$$

Clearly, $Z_C(p^*) = Z_C(p)$ and by Rolle's theorem

$$(6.3) \quad Z_C(Dp^*) \leq Z_C(p^*), \quad \text{where } D = \frac{d}{dx}.$$

Now if $q(x) := D(p^*(x))$, then

$$(6.4) \quad x^{n-1}q\left(\frac{1}{x}\right) = h(x) = \sum_{k=0}^{n-1} (n-k)a_k x^k.$$

Thus, by (6.3) and (6.4),

$$Z_C(h) = Z_C(x^{n-1}q\left(\frac{1}{x}\right)) = Z_C(q) \leq Z_C(p^*) = Z_C(p).$$

If $a_0 a_{n-1} = 0$, then we use a standard perturbation argument. Indeed, for all $\epsilon > 0$, sufficiently small, the coefficients $a_k(\epsilon)$ of $p_\epsilon(x) := p(x + \epsilon)$ are nonzero and $Z_C(p_\epsilon) = Z_C(p)$. Set $h_\epsilon(x) = np_\epsilon(x) - xp'_\epsilon(x)$. Thus, by the first part of the proof,

$$Z_C(h_\epsilon) \leq Z_C(p_\epsilon) = Z_C(p).$$

Since $\epsilon > 0$ was arbitrary, (6.2) follows.

Now to prove (6.1), consider an arbitrary real polynomial $f(x)$ of degree n . For any $\zeta \in \mathbb{R}$, let $p(x) = f(x + \zeta)$. Thus, if

$$h(x) = T_0[p](x) = nf(x + \zeta) - xf'(x + \zeta),$$

then by (6.2)

$$(6.5) \quad Z_C(h) \leq Z_C(f(x + \zeta)) = Z_C(f).$$

Since $h(x - \zeta) = T_\zeta[f](x)$ (and $Z_C(h) = Z_C(h(x - \zeta))$), (6.1) is an immediate consequence of (6.5).

Remark 6.11. It is clear that the differential operator $D = \frac{d}{dx}$, does not increase the number of nonreal zero of a polynomial. Theorem 6.9 asserts that $\forall \zeta \in \mathbb{R}$ the nonlinear differential operator $n + (\zeta - x)D$ does not increase the number of nonreal zeros of a polynomial.

Corollary 6.12. *If $f(x)$ is a real polynomial of degree n with only real zeros, then for any $\zeta \in \mathbb{R}$, $T_\zeta[f](x)$ has only real zeros.*

Proof. By assumption $Z_C(f(x)) = 0$. Hence by Theorem 6.9 (cf. (6.1))

$$Z_C(T_\zeta[f](x)) = 0;$$

that is, $T_\zeta[f](x)$ has only real zeros. \square

6.1. The Polar Derivative Analog of the Laguerre Expression.

Definition 6.13. For the polynomial $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$ the polar derivative analog of the Laguerre expression is

$$M_\zeta[f(z)] := (T_\zeta[f(z)])^2 - f(z)T_\zeta[T_\zeta[f(z)]] = (\zeta - z)^2[(f'(z))^2 - f''(z)f(z)] + n(f(z))^2 + 2(\zeta - z)f'(z)f(z)$$

for all $\zeta \in \mathbb{R}$ and $\zeta \neq -\frac{a_{n-1}}{a_n}$. (See the Appendix for the details of the calculation.)

Theorem 6.14. (Polar Derivative Laguerre Inequality). *Let $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ be a real polynomial of degree n with only real zeros. Then*

$$M_\zeta[f](x) := (T_\zeta[f](x))^2 - f(x)T_\zeta[T_\zeta[f]](x) = (\zeta - x)^2[(f'(x))^2 - f''(x)f(x)] + n(f(x))^2 + 2(\zeta - x)f'(x)f(x) \geq 0, \quad \forall \zeta, x \in \mathbb{R}$$

and $\zeta \neq -\frac{a_{n-1}}{a_n}$. Moreover, equality holds if and only if $f(x) = (x - \zeta)^n$.

Proof. Suppose $f(x) = \prod_{k=1}^n (x - x_k)$ where each $x_k \in \mathbb{R}$. Fix x and $\zeta \in \mathbb{R}$. For now suppose that $x \neq x_k$ for all k . Then

$$\frac{M_\zeta[f](x)}{(f(x))^2} = (\zeta - x)^2 \left[\frac{(f'(x))^2 - f''(x)f(x)}{(f(x))^2} \right] + n + 2(\zeta - x) \frac{f'(x)}{f(x)}.$$

Since

$$\begin{aligned} (\zeta - x)^2 \left[\frac{(f'(x))^2 - f''(x)f(x)}{(f(x))^2} \right] &= (\zeta - x)^2 \left[- \frac{f''(x)f(x) - (f'(x))^2}{(f(x))^2} \right] \\ &= (\zeta - x)^2 \sum_{k=1}^n \frac{1}{(x - x_k)^2} \end{aligned}$$

and

$$(\zeta - x) \frac{f'(x)}{f(x)} = (\zeta - x) \sum_{k=1}^n \frac{1}{x - x_k},$$

it follows that

$$\begin{aligned} \frac{(T_\zeta[f](x))^2 - f(x)T_\zeta[T_\zeta[f]](x)}{(f(x))^2} &= \\ (\zeta - x)^2 \sum_{k=1}^n \frac{1}{(x - x_k)^2} + (\zeta - x) \sum_{k=1}^n \frac{1}{x - x_k} + n &= \\ \sum_{k=1}^n \left[\left(\frac{\zeta - x}{x - x_k} \right)^2 + 2 \frac{\zeta - x}{x - x_k} + 1 \right]. \end{aligned}$$

Setting $\frac{\zeta - x}{x - x_k} = w_k$ we have,

$$\frac{M_\zeta[f](x)}{(f(x))^2} = \sum_{k=1}^n (w_k^2 + 2w_k + 1) = \sum_{k=1}^n (w_k + 1)^2 \geq 0.$$

Since ζ and x were fixed, the above holds for any ζ and any $x \neq x_k$. Since $(f(x))^2 > 0$ it follows that

$$M_\zeta[f](x) = (T_\zeta[f](x))^2 - f(x)T_\zeta[T_\zeta[f]](x) \geq 0.$$

If $x = x_k$ then,

$$(\zeta - x)^2 [(f'(x))^2 - f''(x)f(x)] + n(f(x))^2 + 2(\zeta - x)f'(x)f(x) = (\zeta - x)^2 (f'(x))^2 \geq 0.$$

Since

$$M_\zeta[f](x) = \sum_{k=1}^n \left(\frac{\zeta - x_k}{x - x_k} \right)^2 = 0$$

if and only if $\zeta = x_k$ for all k . Equality holds only when $f(x) = (x - \zeta)^n$. \square

Remark 6.15. We note here that the converse of Theorem 6.14 is not true. Consider the following counterexample. Let $f(x) = x^4 - 1$. A calculation shows that

$$M_\zeta[f](x) = 4(\zeta^2 x^6 + 3x^4 - 8\zeta x^3 + 3\zeta^2 x^2 + 1).$$

For $\zeta \leq 0$ and $x \geq 0$ all of the terms are positive, hence $M_\zeta[f](x) \geq 0$. Also for $\zeta > 0$ and $x < 0$ the terms are all positive, hence $M_\zeta[f](x) \geq 0$. By symmetry it suffices to show that $M_\zeta[f](x) \geq 0$ for $\zeta > 0$ and $x > 0$. Suppose that $\zeta > 0$ and $x > 0$. Fix $x > 0$. Now, consider the quadratic polynomial in ζ ,

$$(6.6) \quad \zeta^2 [x^6 + 3x^2] + \zeta [-8x^3] + 3x^4 + 1.$$

By the quadratic formula the zeros of this quadratic (6.6) are

$$\begin{aligned}\zeta &= \frac{8x^3 \pm \sqrt{64x^6 - 4(x^6 + 3x^2)(3x^4 + 1)}}{2x^6 + 6x^2} \\ &= \frac{8x^3 \pm \sqrt{24x^6 - 12x^{10} - 12x^2}}{2x^6 + 6x^2} \\ &= \frac{8x^3 \pm 2x\sqrt{3}\sqrt{-8x^8 + 2x^4 - 1}}{2x^6 + 6x^2}.\end{aligned}$$

Since $-8x^8 + 2x^4 - 1 = -(x^4 - 1)^2 \leq 0$ for any x . We can conclude that $\zeta^2[x^6 + 3x^2] + \zeta[-8x^3] + 3x^4 + 1 \geq 0$ for all $x > 0$ and all $\zeta > 0$. Therefore, $M_\zeta[f](x) \geq 0$ for all x and ζ . Thus, if $f(x) = x^4 - 1$, then $f \notin \mathcal{L} - \mathcal{P}$, but $M_\zeta[f](x) \geq 0$ for all $x, \zeta \in \mathbb{R}$.

Remark 6.16. We next show by means of an example that if $L[f](x) \geq 0 \forall x \in \mathbb{R}$, then, in general, $M_\zeta[f](x)$ need not be nonnegative for $x, \zeta \in \mathbb{R}$. This example illustrates the following statement,

$$L[f](x) \geq 0 \quad \forall x \in \mathbb{R}, \quad \not\Rightarrow \quad M_\zeta[f](x) \geq 0 \quad \forall x, \zeta \in \mathbb{R},$$

where $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ and $\zeta \neq -\frac{a_{n-1}}{a_n}$.

Consider the following example. For the polynomial $f(x) = x^4 + 3x^2 + 20x + 2$, the Laguerre expression is,

$$L[f](x) = 2(2x^6 + 3x^4 - 40x^3 - 3x^2 + 60x + 194) \geq 0 \quad \forall x \in \mathbb{R},$$

and the polar derivative analog of the Laguerre expression at $\zeta = 1$ is,

$$\begin{aligned}M_1[f](x) &= -2(x^6 + 48x^5 - 180x^4 - 112x^3 - 63x^2 - 204x - 242) < 0 \\ &\text{for } x \in (-\infty, -51.46 \dots) \cup (4.55 \dots, \infty).\end{aligned}$$

Remark 6.16 motivates the following proposition.

Proposition 6.17. *Suppose $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ is a real polynomial and*

$$M_\zeta[f](x) \geq 0 \quad \forall x \forall \zeta \in \mathbb{R}, \quad \zeta \neq -\frac{a_{n-1}}{a_n}.$$

Then

$$L[f](x) \geq 0 \quad \forall x \in \mathbb{R}.$$

Proof. Suppose that $M_\zeta[f](x) \geq 0$ for all $x, \zeta \in \mathbb{R}$. For a fixed, but arbitrary, $x \in \mathbb{R}$, consider

$$\lim_{\zeta \rightarrow \infty} \frac{M_\zeta[f](x)}{\zeta^2} = \lim_{\zeta \rightarrow \infty} L[f](x) - \frac{n(f(x))^2 + 2(\zeta - x)f'(x)f(x)}{\zeta^2} = L[f](x).$$

Thus, $L[f](x) \geq 0$. Since x was arbitrary, we conclude that $L[f](x) \geq 0$ for all $x \in \mathbb{R}$. \square

Remark 6.18. The above proposition says that the polar Laguerre inequality is *stronger* than the Laguerre inequality.

Remark 6.19. We may use the Laguerre inequality to prove a simple fact from the calculus. Let $f(x)$ be a real polynomial. If $f'(x_0) = 0$, $f(x_0) > 0$, and $f''(x_0) > 0$; that is, if f has a positive local minimum, then $L[f](x_0) < 0$, which implies that f has a nonreal zero. We next show that a similar result holds for the polar derivative version of the Laguerre inequality.

Proposition 6.20. *Let $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$. If $T_\zeta[f](x_0) = 0$, $f(x_0) > 0$, and $T_\zeta^2[f](x_0) > 0$ for all $\zeta \in \mathbb{R}$ and $\zeta \neq -\frac{a_{n-1}}{a_n}$, then f has a nonreal zero.*

Proof. Since $T_\zeta[f](x_0) = 0$, $f(x_0) > 0$, and $T_\zeta^2[f](x_0) > 0$ for all $\zeta \in \mathbb{R}$ we have,

$$M_\zeta[f](x_0) = (T_\zeta[f](x_0))^2 - f(x_0)T_\zeta^2[f](x_0) < 0.$$

Therefore, by Theorem 6.14, f must have a nonreal zero. \square

Remark 6.21. If $f(x)$ has only real zeros then, by Corollary 6.12, $T_\zeta[f](x)$ has only real zeros for all $\zeta \in \mathbb{R}$. Consequently, if $f(x)$ has only real zeros, then $M_\zeta[T_{\zeta_1}[f]](x) \geq 0$ for all $x, \zeta, \zeta_1 \in \mathbb{R}$ with the possible exception of ζ and ζ_1 being the negative ratio of the n^{th} and $(n-1)^{\text{st}}$ coefficients of their respective polynomials.

Indeed, let $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$. By Lemma 6.1 we have,

$$\begin{aligned} T_{\zeta_1}[f](x) &= \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + \zeta_1 a_{k+1}) x^k \\ &:= \sum_{k=0}^{n-1} \binom{n-1}{k} a_k^{(1)} x^k, \\ T_{\zeta_2} T_{\zeta_1}[f](x) &= \sum_{k=0}^{n-1} \binom{n-2}{k} (a_k^{(1)} + \zeta_2 a_{k+1}^{(1)}) x^k \\ &:= \sum_{k=0}^{n-2} \binom{n-2}{k} a_k^{(2)} x^k \\ &\vdots \\ T_{\zeta_m} \cdots T_{\zeta_1}[f](x) &= \sum_{k=0}^{n-m} \binom{n-m}{k} (a_k^{(m-1)} + \zeta_m a_{k+1}^{(m-1)}) x^k \\ &:= \sum_{k=0}^{n-m} \binom{n-m}{k} a_k^{(m)} x^k. \end{aligned}$$

If $f(x)$ has only real zeros and $\zeta_j \neq -\frac{a_{n-j}^{(j-1)}}{a_{n-j+1}^{(j-1)}}$, then by Theorem 6.14

$$M_{\zeta_m}[T_{\zeta_{m-1}} \cdots T_{\zeta_1}](x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

6.2. A Necessary and Sufficient Condition. In [7, Theorem 2.12] the authors state a necessary and sufficient condition for a real entire function to be in the Laguerre-Pólya class, $\mathcal{L} - \mathcal{P}$. The following theorem is a polar derivative analog of this theorem for real polynomials.

Theorem 6.22. *Suppose $p(z)$ is a real polynomial of degree n with $p(z) = \prod_{k=1}^n (z - z_k)$. Then the polynomial p has only real zeros if and only if,*

$$\Im \left(\frac{T_\zeta[p](z)}{p(z)} \right) < 0 \quad \text{for all } \zeta > \max_{1 \leq k \leq n} \Re z_k \text{ and } z \in H^+,$$

where $H^+ = \{z \in \mathbb{C} : \Im z > 0\}$.

Proof. Let $p(z) = \prod_{k=1}^n (z - z_k)$, set $z = x + iy$ and $z_k = a_k + ib_k$.

First, compute the imaginary part of $\frac{T_\zeta[p](z)}{p(z)}$,

$$\begin{aligned}
\frac{T_\zeta[p](z)}{p(z)} &= \frac{np(z) + (\zeta - z)p'(z)}{p(z)} \\
&= n + (\zeta - z) \sum_{k=1}^n \frac{1}{z - z_k} \\
&= n + \sum_{k=1}^n \frac{\zeta - x - iy}{(x - a_k) + i(y - b_k)} \\
&= n + \sum_{k=1}^n \frac{(\zeta - x)(x - a_k) - (y - b_k)y}{(x - a_k)^2 + (y - b_k)^2} - i \sum_{k=1}^n \frac{(\zeta - x)(y - b_k) + y(x - a_k)}{(x - a_k)^2 + (y - b_k)^2}.
\end{aligned}$$

Thus,

$$\Im\left(\frac{T_\zeta[p](z)}{p(z)}\right) = - \sum_{k=1}^n \frac{(\zeta - x)(y - b_k) + y(x - a_k)}{(x - a_k)^2 + (y - b_k)^2}.$$

Suppose that $\Im(z) > 0$, all the zeros of p are real ($b_k = 0$ for all k) and that $\zeta > a_k$ for all k . Then,

$$\begin{aligned}
\Im\left(\frac{T_\zeta[p](z)}{p(z)}\right) &= - \sum_{k=1}^n \frac{(\zeta - x)y - y(x - a_k)}{(x - a_k)^2 + y^2} \\
&= - \sum_{k=1}^n \frac{y(\zeta - a_k)}{(x - a_k)^2 + y^2} < 0.
\end{aligned}$$

Conversely suppose that,

$$\Im\left(\frac{T_\zeta[p](z)}{p(z)}\right) = - \sum_{k=1}^n \frac{(\zeta - x)(y - b_k) + y(x - a_k)}{(x - a_k)^2 + (y - b_k)^2} < 0$$

for all $z \in H^+$ and $\zeta > \max a_k$, where $a_k = \Re(z_k)$. We now use an argument by contradiction. Suppose that $b_1 > 0$ and $b_2 = -b_1$; that is, there is a nonreal zero at z_1 and its conjugate is z_2 so $a_1 = a_2$. Define

$$R(z) = R(x + iy) = \Im\left(\frac{T_\zeta[p](x + iy)}{p(x + iy)}\right).$$

Fix $\epsilon \in \mathbb{R}$ such that $1 > \epsilon > 0$. Then

$$\begin{aligned}
R(a_1 + ib_1(1 - \epsilon)) &= - \sum_{k=1}^n \frac{b_1(1 - \epsilon)(a_1 - a_k) + (b_1(1 - \epsilon) - b_k)(\zeta - a_1)}{(a_1 - a_k)^2 + (b_1(1 - \epsilon) - b_k)^2} \\
&= - \left[\frac{b_1(1 - \epsilon)(a_1 - a_1) + (b_1(1 - \epsilon) - b_1)(\zeta - a_1)}{(a_1 - a_1)^2 + (b_1(1 - \epsilon) - b_1)^2} \right] \\
&\quad - \left[\frac{b_1(1 - \epsilon)(a_1 - a_2) + (b_1(1 - \epsilon) - b_2)(\zeta - a_1)}{(a_1 - a_2)^2 + (b_1(1 - \epsilon) - b_2)^2} \right] \\
&\quad - \sum_{k=3}^n \frac{b_1(1 - \epsilon)(a_1 - a_k) + (b_1(1 - \epsilon) - b_k)(\zeta - a_1)}{(a_1 - a_k)^2 + (b_1(1 - \epsilon) - b_k)^2} \\
&= - \left[\frac{-\epsilon b_1(\zeta - a_1)}{(-\epsilon b_1)^2} \right] - \left[\frac{(2b_1 - \epsilon b_1)(\zeta - a_1)}{(2b_1 - \epsilon b_1)^2} \right] \\
&\quad - \sum_{k=3}^n \frac{b_1(1 - \epsilon)(a_1 - a_k) + (b_1(1 - \epsilon) - b_k)(\zeta - a_1)}{(a_1 - a_k)^2 + (b_1(1 - \epsilon) - b_k)^2}.
\end{aligned}$$

Thus, $R(a_1 + ib_1(1 - \epsilon)) = \frac{\zeta - a_1}{\epsilon - b_1} + O(1)$ as $\epsilon \rightarrow 0$. Since $\zeta > \max a_k$ and $b_1 > 0$, $R(a_1 + ib_1(1 - \epsilon)) \rightarrow \infty$ as $\epsilon \rightarrow 0$. Hence, if $p(z)$ has a nonreal zero, then we can find a z with $\Im(z) > 0$ such that $R(z) > 0$, a contradiction. This completes the proof of the theorem. \square

Remark 6.23. The following proposition gives a relationship between the generalized center of mass of the zeros of a polynomial and the polar derivative.

Proposition 6.24. *Let f be a real polynomial. Then $T_\zeta[f](x) = 0$ for some $x, \zeta \in \mathbb{R} \iff \zeta$ is the center of mass of the zeros of f with respect to the point x .*

Proof. Suppose that $T_\zeta[f](x) = 0$ for some $x, \zeta \in \mathbb{R}$. Then,

$$\zeta = x - n \frac{f(x)}{f'(x)}$$

which is, by definition, the center of mass of the zeros of f with respect to the point at x (cf. Section 5).

Suppose that ζ is the center of mass of the zeros of f with respect to the point at x . Then,

$$\zeta = x - n \frac{f(x)}{f'(x)}.$$

Computing $T_\zeta[f](x)$ with this value of ζ yields, $T_\zeta[f](x) = 0$. \square

7. THE HAWAII CONJECTURE

A problem due to Gauss, dates back to c. 1836, asked to find a relationship between the number of real zeros of $Q(x) = \left(\frac{p'(x)}{p(x)} \right)'$ and the number of nonreal zeros of the real polynomial $p(x)$. In 1987, in connection with their solution to the Pólya-Wiman conjecture (i.e., $f \in \mathcal{L} - \mathcal{P}^*$ and if f has order less than 2, then $f^{(n)} \in \mathcal{L} - \mathcal{P}$ for all n sufficiently large), Craven, Csordas, and Smith [4] conjectured a precise upper bound for the number of real zeros of $Q(x)$ (cf. Conjecture 7.1 below). We mention here parenthetically that Sheil-Small [1, xvi and Chapter 9] attributes the appellation ‘‘Hawaii Conjecture’’ (Conjecture 7.1) to Eremenko [9].

According to Sheil-Small, Eremenko nicknamed Conjecture 7.1 the “Hawai’i Conjecture” because the three authors are all from the University of Hawai’i.

Conjecture 7.1. (The Hawai’i Conjecture [4]). *If $p(x)$ is a real polynomial of degree $n \geq 2$ with $2d$ nonreal zeros, $0 \leq 2d \leq n$. Then*

$$Z_R(Q(x)) = Z_R\left(\frac{p'(x)}{p(x)}\right)' \leq 2d,$$

where $Z_R(Q)$ denotes the number of real zeros of the rational function Q , counting multiplicities.

By the Laguerre inequality we know that when $p(x)$ has only real zeros then $Q(x) < 0$ and hence $Q(x)$ has no real zeros. It is also known that when $p(x)$ has exactly two nonreal zeros, then $Z_R(Q(x)) \leq 2$ (see Section 7.2). In the next subsection we first show that the conjecture is valid when $p'(x)$ has only real simple zeros (cf. [4, Theorem 1]).

7.1. The Case of Simple Real Zeros of the Derivative.

Theorem 7.2. (A Special Case of the Hawai’i Conjecture). *Let $p(x)$ be a real polynomial of degree $n \geq 2$ with $2d$ nonreal zeros ($0 \leq 2d \leq n$). Suppose that $p'(x)$ has only real simple zeros. Then $Q(x) = \left(\frac{p'(x)}{p(x)}\right)'$ has exactly $2d$ real zeros.*

The proof of Theorem 7.2 will be based on the following lemmas.

Lemma 7.3. *Let $p(x)$ be a real polynomial of degree $n \geq 2$. Suppose that $p'(x)$ has only real simple zeros:*

$$t_1 < t_2 < \cdots < t_{n-1}.$$

Define the following intervals,

$$\begin{aligned} I_j &= (t_j, t_{j+1}), \quad j = 1, 2, \dots, n-2 \\ I_{-\infty} &= (-\infty, t_1), \text{ and} \\ I_{\infty} &= (t_{n-1}, \infty). \end{aligned}$$

Then

- (a) *$p(x)$ has at most one real zero in any one of these intervals and,*
- (b) *for any $\lambda \in \mathbb{R}$ the polynomial $p(x) + \lambda p'(x)$ has at most two real zeros (counting multiplicity) in any one of these intervals.*

For the intervals defined in Lemma 7.3, we define an interval to be of *Type 1*, if it contains a zero of $p(x)$ and to be of *Type 2* if it does not contain a zero of $p(x)$. If t_k is a multiple zero of $p(x)$ then we say that both the intervals I_k and I_{k-1} are intervals of Type 1.

Lemma 7.4. *Let $p(x)$ be a real polynomial with degree $n \geq 2$. Suppose that $p'(x)$ has only real simple zeros and $p(x)$ has $2d$ ($0 \leq 2d \leq n$) nonreal zeros. Then there are exactly $2d$ intervals, I_k , of Type 2.*

Lemma 7.5. *Let $p(x)$ be a real polynomial of degree $n \geq 2$ with $2d$ ($0 \leq 2d \leq n$) nonreal zeros. Let $p'(x)$ have only real simple zeros. Then*

$$L[p](x) = (p'(x))^2 - p(x)p''(x)$$

has no zeros in an interval of Type 1 and exactly one real zero in each interval of Type 2.

Proof of Lemma 7.3. (a) Suppose for the sake of argument that $p(x)$ has two real zeros x_1 and x_2 in some interval I_k , $k = \pm\infty, 1, 2, \dots, n-1$. Then by Rolle's Theorem, $p'(x)$ must have at least one zero in the interval (x_1, x_2) . But, this contradicts the construction of I_k . Therefore, $p(x)$ has at most 1 real zero in I_k .

Note that it is still possible for $p(x)$ to have no zeros in the I_k , consider, for example, $p(x) = x^4 - 2x^2 + 3$.

(b) For $\lambda = 0$ we are done by part (a). Suppose that $\lambda \neq 0$ and let $\alpha = \frac{1}{\lambda}$. Consider

$$\alpha p'(x) + p''(x) = e^{-\alpha x} D[e^{\alpha x} p'(x)],$$

where D is differentiation with respect to x . By Rolle's theorem $\alpha p'(x) + p''(x)$ has only real simple zeros (since $p'(x)$ has only real simple zeros).

If $\alpha p'(x) + p''(x)$ has more than one real zero in any I_k , then either $e^{\alpha x} p'(x)$ has another real zero in I_k or $e^{\alpha x} p'(x)$ has a nonreal zero. Hence, $\alpha p'(x) + p''(x)$ has at most one real zero in I_k .

Since $\alpha p'(x) + p''(x)$ has at most one real zero in I_k and $D[\alpha p(x) + p'(x)] = \alpha p'(x) + p''(x)$, the polynomial $\alpha p(x) + p'(x)$ has at most 2 zeros in I_k . To see this, suppose that $\alpha p(x) + p'(x)$ has more than two (say three or more) zeros in I_k , then by Rolle's theorem $\alpha p'(x) + p''(x)$ would have at least two zeros in I_k , contradicting the above argument that $\alpha p'(x) + p''(x)$ has at most one zero in I_k . Since $\lambda[\alpha p(x) + p'(x)] = p(x) + \lambda p'(x)$ we conclude that $p(x) + \lambda p'(x)$ has at most two zeros in any I_k . \square

Proof of Lemma 7.4. Since the intervals of Type 1 have exactly one zero of $p(x)$, the number of Type 1 intervals is exactly $n - 2d$. Since $p'(x)$ has only simple zeros there are n intervals (as defined in Lemma 7.3). Therefore the number of Type 2 intervals is $n - (n - 2d) = 2d$. \square

Proof of Lemma 7.5. Consider the rational function

$$\varphi(x) = \frac{p(x)}{p'(x)}.$$

Then

$$\varphi'(x) = \frac{(p'(x))^2 - p(x)p''(x)}{(p'(x))^2}.$$

In order to prove Lemma 7.5 it will suffice to show that $\varphi'(x)$ has no zeros in an interval of Type 1 and exactly one zero in each interval of Type 2.

Case 1: Let I_k be an interval of Type 1 and $\lambda \in \mathbb{R}$. Then

$$\varphi(x) + \lambda = \frac{p(x) + \lambda p'(x)}{p'(x)}$$

has an odd number of zeros in I_k . To see this, suppose that $\varphi(x) + \lambda$ has an even number of zeros in I_k . Then either $\varphi(x)$ has no zeros in I_k or φ has an even number of zeros (counting multiplicity) in I_k . Thus, $p(x)$ has either an even number of zeros or no zeros in I_k . This contradicts part (a) of Lemma 7.3 (which says that p has at most one zero in a Type 1 interval). Now by Lemma 7.3

$$\varphi(x) + \lambda = \frac{p(x) + \lambda p'(x)}{p'(x)}$$

has at most two zeros in I_k . But the only odd integer less than two is one, so $\varphi(x) + \lambda$ has exactly one zero (counting multiplicity) in I_k , for any $\lambda \in \mathbb{R}$. Thus, either $\varphi'(x) > 0$ or $\varphi'(x) < 0$ on I_k , and $\varphi'(x) \neq 0$ on I_k .

Case 2: Let I_k be an interval of Type 2. Since $p(x) \neq 0$ on I_k , either $\varphi(x) > 0$ or $\varphi(x) < 0$ on I_k . Suppose that $\varphi(x) > 0$ on I_k , then $\varphi(x)$ must attain its absolute minimum on I_k . Since $\varphi(x) > 0$ on I_k and $x = t_k$ and $x = t_{k+1}$ are vertical asymptotes, we know that $\lim_{x \rightarrow t_k^+} \varphi(x) = +\infty$ and $\lim_{x \rightarrow t_{k+1}^-} \varphi(x) = +\infty$. Let $s_0 \in I_k$ be the absolute minimum of φ on I_k and set $\lambda_0 = \varphi(s_0)$. For $\lambda < \lambda_0$, the function $\varphi(x) - \lambda$ has no zeros in I_k . For $\lambda > \lambda_0$, the function $\varphi(x) - \lambda$ has at least two zeros in I_k . To see this, suppose that $\varphi(x) - \lambda$ has exactly one zero in I_k . Then either

$$\lim_{x \rightarrow t_k^+} \varphi(x) = +\infty \text{ and } \lim_{x \rightarrow t_{k+1}^-} \varphi(x) = -\infty$$

or

$$\lim_{x \rightarrow t_k^+} \varphi(x) = -\infty \text{ and } \lim_{x \rightarrow t_{k+1}^-} \varphi(x) = +\infty$$

which is a contradiction to the construction of $\varphi(x)$. Hence $\varphi(x) - \lambda$ has at least 2 zeros in I_k . But, by Lemma 7.3 for $\lambda > \lambda_0$, $\varphi(x) - \lambda$ has at most two zeros in I_k . Hence $\varphi(x) - \lambda$ has exactly two zeros in I_k . Suppose that $\varphi(s_0) = \lambda$, then since s_0 is the absolute minimum it is also a zero of $\varphi'(x)$. Hence, s_0 is a multiple zero of $\varphi(x) - \lambda_0$. Therefore, for $\lambda \geq \lambda_0$ (and $\varphi(x) > 0$) the function $\varphi(x) - \lambda$ has exactly two real zeros in I_k . By Rolle's theorem the function $\varphi'(x)$ has at least one zero in I_k . A similar argument holds for $\varphi(x) < 0$ on I_k . Now, suppose that $\varphi'(x)$ has more than two zeros in I_k . Then for a sufficiently large λ , the function $\varphi(x) - \lambda$ will have more than two zeros in I_k , a contradiction. Therefore we can conclude that $\varphi'(x)$ has exactly one zero in I_k , where I_k is an interval of Type 2. \square

Proof of Theorem 7.2. From Lemmas 7.3, 7.4 and 7.5 we infer that the number of real zeros of $L[p](x)$ is precisely $2d$. Therefore, the number real of zeros of $Q(x) = \left(\frac{p'(x)}{p(x)}\right)'$ is precisely $2d$, as desired. \square

7.2. The Hawai'i Conjecture for Polynomials With Exactly Two Nonreal Zeros. We begin this section with a remarkably simple proof which shows that the Hawai'i Conjecture is valid when $p(x)$ is a polynomial of the form

$$p(x) = ((x - x_0)^2 + y_0^2)(x - \beta)^n, \quad x_0, y_0, \beta \in \mathbb{R}, \quad n = 0, 1, 2, \dots$$

It is clear that we may assume, via a change of variables, that $p(x)$ is of the form

$$(7.1) \quad p(x) = (x^2 + 1)(x - \alpha)^n \quad (n = 0, 1, 2, \dots).$$

Proposition 7.6. *Let $p(x)$ be a polynomial of the form (7.1). Then*

$$(7.2) \quad Z_R\left(\left(\frac{p'}{p}\right)'\right) \leq 2,$$

where $Z_R(Q)$ denotes the number of real zeros of the rational function Q , counting multiplicity.

Proof. We first consider the case when $n = 1$ (the case when $n = 0$ is trivial). Then

$$\left(\frac{p'}{p}\right)'(x) = \frac{2(1 - x^2)}{(1 + x^2)^2} - \frac{1}{(x - \alpha)^2} = \frac{h_1(x)}{(1 + x^2)^2(x - \alpha)^2},$$

where $h_1(x) = -1 + 2\alpha^2 - 4\alpha x - 2\alpha^2 x^2 + 4\alpha x^3 - 3x^4$. Then via the translation $x \rightarrow x + \frac{\alpha}{3}$ we obtain the 4th degree polynomial

$$(7.3) \quad h_1\left(x + \frac{\alpha}{3}\right) = -\frac{1}{9}(\alpha^2 - 3)^2 - \frac{4\alpha}{9}(\alpha^2 + 9)x - 3x^4.$$

If $\alpha^2 = 3$, then $h_1(x + \frac{\alpha}{3})$ has 2 nonreal zeros. If $\alpha^2 \neq 3$, then, by Fact 1.5 of Section 1, (7.3) has 2 nonreal zeros (and whence at most 2 real zeros) so that (7.2) holds.

Next suppose that $n \geq 2$. Then

$$\left(\frac{p'}{p}\right)'(x) = \frac{2(1-x^2)}{(1+x^2)^2} - \frac{n}{(x-\alpha)^2} = \frac{h_n(x)}{(1+x^2)^2(x-\alpha)^2},$$

where $h_n(x) = 2\alpha^2 - n - 4\alpha x + 2(1 - \alpha^2 - n) + 4\alpha x^3 - (2+n)x^4$. Hence,

$$h_n''(x) = -4[\alpha^2 + n - 1 - 6\alpha x + (6+3n)x^2].$$

Since the discriminant of this quadratic is

$$-3(n-1)(2+\alpha^2+n) < 0 \quad \text{for } n \geq 2 \text{ and all } \alpha \in \mathbb{R},$$

we conclude that $h_n''(x)$ has 2 nonreal zeros and whence $h_n(x)$ has at least 2 nonreal zeros. \square

The proof of the Hawai'i Conjecture for polynomials with exactly two nonreal zeros and no restriction on the real zeros requires the use of the following lemma.

Lemma 7.7. *Let $p(x)$ be a real polynomial of the form*

$$p(x) = (x^2 + 1)q(x),$$

where $q(x) = \prod_{k=1}^n (x - a_k)$ and $a_k \in \mathbb{R}$ for all k . If $\alpha, \beta \in (-1, 1)$ are zeros of

$$Q(x) = \left(\frac{p'(x)}{p(x)}\right)',$$

with $\alpha < \beta$, then $a_k \notin (\alpha, \beta)$ for all k .

Proof. Suppose that α and β , with $-1 < \alpha < \beta < 1$, are two zeros of $Q(x)$. Assume that there is a zero of p , $a = a_k$, such that $\alpha < a < \beta$. Since,

$$\sum_{k=1}^n \frac{1}{(\alpha - a_k)^2} = \frac{2(1 - \alpha^2)}{(1 + \alpha^2)^2}$$

we get

$$\frac{1}{(\alpha - a)^2} \leq \frac{2(1 - \alpha^2)}{(1 + \alpha^2)^2}.$$

This is equivalent to

$$\frac{1 + \alpha^2}{\sqrt{2(1 - \alpha^2)}} \leq a - \alpha.$$

Similarly for β we get,

$$\frac{1 + \beta^2}{\sqrt{2(1 - \beta^2)}} \leq \beta - a.$$

From the above two equations it follows that

$$\frac{1 + \beta^2}{\sqrt{2(1 - \beta^2)}} + \frac{1 + \alpha^2}{\sqrt{2(1 - \alpha^2)}} \leq \beta - \alpha.$$

Hence,

$$(7.4) \quad \alpha + \frac{1 + \alpha^2}{\sqrt{2(1 - \alpha^2)}} \leq \beta - \frac{1 + \beta^2}{\sqrt{2(1 - \beta^2)}}.$$

We now prove the following inequality for $-1 < x < 1$,

$$(7.5) \quad |x| < \frac{1 + x^2}{\sqrt{2(1 - x^2)}}.$$

Since $-1 < x < 1$ we know that

$$0 < 3x^4 + 1, \quad 2x^2(1 - x^2) < (x^2 + 1)^2, \quad \text{and} \quad |x|\sqrt{2(1 - x^2)} < x^2 + 1.$$

Therefore, inequality (7.5) holds.

Since $-1 < \alpha < \beta < 1$, we can use inequality (7.5) (with $\alpha = |\alpha|$ and $\beta = |\beta|$) to get,

$$\begin{aligned} \alpha + \frac{1 + \alpha^2}{\sqrt{2(1 - \alpha^2)}} &> 0 \\ \beta - \frac{1 + \beta^2}{\sqrt{2(1 - \beta^2)}} &< 0. \end{aligned}$$

This contradicts (7.4) and, hence there can be no real zero, a , of p that lies between the two zeros of $Q(x)$, α and β . \square

In [17] Sheil-Small proves the following result.

Proposition 7.8. (The Hawai'i Conjecture for Polynomials with Exactly Two Nonreal Zeros [17, p. 308]) *Let $p(z) = (z^2 + 1)q(z)$, where q has exactly $n - 2$ real zeros. Then $(\frac{p'}{p})'$ has at most two real zeros.*

Rather than reproducing the proof given in [17], we present here, in conjunction with a calculation, a heuristic geometric argument. The proof in [17, p. 309] utilizes the Riemann-Hurwitz Formula [17, p. 353] which lies outside the scope of this paper.

Suppose that $q(z) = \prod_{k=1}^{n-2} (z - a_k)$. Then

$$\begin{aligned} (\log p(z))' &= \frac{d}{dz} \left(\log(z^2 + 1) + \sum_{k=1}^{n-2} \log(z - a_k) \right) \\ &= \frac{2z}{(z^2 + 1)} + \sum_{k=1}^{n-2} \frac{1}{z - a_k} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{p'(z)}{p(z)} \right)' &= \frac{(z^2 + 1)2 - 2z(2z)}{(z^2 + 1)^2} - \sum_{k=1}^{n-2} \frac{1}{(z - a_k)^2} \\ &= \frac{2(1 - z^2)}{(z^2 + 1)^2} - \sum_{k=1}^{n-2} \frac{1}{(z - a_k)^2}. \end{aligned}$$

It suffices to show that

$$(7.6) \quad \sum_{k=1}^{n-2} \frac{1}{(x - a_k)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2}$$

has at most two real solutions in the interval $-1 < x < 1$, since the right side is always positive and the left side is only positive for x between -1 and 1 .

Set $g(x) = \sum_{k=1}^{n-2} \frac{1}{(x-a_k)^2}$. Then, $g''(x) = 6 \sum_{k=1}^{n-2} \frac{1}{(x-a_k)^4} > 0$ and $g(x)$ is concave up on the real line. Considering the graph of $\frac{2(1-x^2)}{(x^2+1)^2}$ and the possible intersections with the graph of $g(x)$, we can see that the number of intersection will be 0, 1, or 2, provided that there are no asymptotes between any two intersections; i.e., no zeros of $p(x)$ between any two intersection points. By Lemma 7.7 there are no zeros of $p(x)$ between any two solutions to the above equation (7.6). Hence, there can be at most two real zeros of $\left(\frac{p'(x)}{p(x)}\right)'$.

Remark 7.9. In [17] the author also proves that the Hawai'i Conjecture is true for polynomials of degree 4, 6, 8, and 10 that have nonreal zeros that are purely imaginary.

8. APPLICATIONS OF THE LAGUERRE INEQUALITIES

In the theory of special functions there are many generating functions which belong to the Laguerre-Pólya class. Our goal in this section is merely to indicate how the forgoing theory can be used to establish a plethora of inequalities involving several families of special functions. In order to motivate our first result here (cf. Example 8.1 below), we first recall that if

$$(8.1) \quad \varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}$$

and

$$(8.2) \quad L_n[\varphi^{(p)}](x) = \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-1)^{k+n}}{(2n)!} \varphi^{(p+k)}(x) \varphi^{(p+2n-k)}(x), \quad n, p = 0, 1, 2, \dots,$$

then the extended Laguerre inequalities (Theorem 2.3) hold:

$$(8.3) \quad L_n[\varphi^{(p)}](x) \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ and } n, p = 0, 1, 2, \dots$$

Theorem 8.1. ([8]). *Let $\varphi(x) \in \mathcal{L} - \mathcal{P}$, where $\varphi(x)$ is defined by (8.1). For $n, p = 0, 1, 2, \dots$, set*

$$(8.4) \quad \sigma_{n,p} := \sum_{k=0}^n \binom{n}{k} (\gamma_{p+k+1} \gamma_{p+n-k+1} - \gamma_{p+k+2} \gamma_{p+n-k}).$$

If $\gamma_k > 0$ for $k = 0, 1, 2, \dots$, then

$$(8.5) \quad \sigma_{n,p} \geq 0 \quad \text{for } n, p = 0, 1, 2, \dots$$

Using Theorem 8.1 we can show that the Laguerre expression, $L_1[\varphi](x)$, is “strongly convex” for $x \geq 0$ in the sense that

$$L_1^{(\nu)}[\varphi](x) \geq 0 \quad \text{for } x \geq 0 \quad \text{and } \nu = 0, 1, 2, \dots$$

Furthermore, the following slightly stronger result holds.

Corollary 8.2. *Let $\varphi(x) \in \mathcal{L} - \mathcal{P}$, where $\varphi(x)$ is defined by (8.1). Suppose that $\gamma_k > 0$ for $k = 0, 1, 2, \dots$. Let*

$$E_p(x) := L_1[\varphi^{(p)}](x) = (\varphi^{(p+1)}(x))^2 - \varphi^{(p)}(x) \varphi^{(p+2)}(x), \quad p = 0, 1, 2, \dots$$

Then

$$(8.6) \quad E_p^{(\nu)}(x) \geq 0 \quad \text{for all } x \geq 0 \quad \text{and} \quad \nu, p = 0, 1, 2, \dots$$

Proof. In order to prove (8.6), it suffices to show that for each fixed nonnegative integer p , the Maclaurin series coefficients of $E_p(x)$ are nonnegative. Since $\varphi(x) \in \mathcal{L} - \mathcal{P}$ the following calculation is readily justified

$$\begin{aligned} E_p(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \gamma_{p+k+1} \gamma_{p+n-k+1} \frac{x^n}{n!} \\ &\quad - \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \gamma_{p+k+2} \gamma_{p+n-k} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \sigma_{n,p} \frac{x^n}{n!}, \end{aligned}$$

where $\sigma_{n,p}$ is defined in (8.4). Since $\sigma_{n,p} \geq 0$ ($n, p = 0, 1, 2, \dots$) the assertion (8.6) follows. \square

The inequalities in (8.5) and (8.6) form a collection of necessary conditions for a real entire function, $\varphi(x)$ (of the form (8.1)), to possess only real zeros. If the Taylor coefficients in (8.1) depend on a parameter t , then we obtain the following result.

Theorem 8.3. *Let $\emptyset \neq E \subseteq \mathbb{R}$. Let $\{u_n(t)\}_{n=1}^{\infty}$ be a sequence of real-valued functions defined on E . Suppose that for each $t \in E$*

$$(8.7) \quad F(x) = F(x; t) = \sum_{n=0}^{\infty} u_n(t) \frac{x^n}{n!} \in \mathcal{L} - \mathcal{P}.$$

Let

$$(8.8) \quad \Delta_{n,p}(t) = \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-1)^{k+n}}{(2n)!} u_{p+k}(t) u_{p+2n-k}(t), \quad p, n = 0, 1, 2, \dots$$

Then

$$(8.9) \quad \Delta_{n,p}(t) \geq 0 \quad \text{for } t \in E \text{ and } n, p = 0, 1, 2, \dots$$

We omit the proof of (8.9) since it follows, *mutatis mutandis*, from our proof of Theorem 2.3.

In [13] Patrick lists the names of several special functions for which inequality (8.9) holds. Now in order to ascertain the validity of (8.9), one must verify that the generating function $F(x) \in \mathcal{L} - \mathcal{P}$ (see (8.7)) for $t \in E$. In the concrete examples below, we refer to Rainville [16] for the definitions of the special functions under consideration.

Example 8.4. (Legendre polynomials $P_n(t)$ [16, p. 165]).

$$(8.10) \quad F(x) = e^{xt} J_0(x\sqrt{1-t^2}) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!},$$

where $E = (-1, 1)$ and

$$(8.11) \quad J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} x^{2k}$$

is the Bessel function of the first kind of index zero [16, p. 109]. Since $J_0(x) \in \mathcal{L} - \mathcal{P}$ [14, p. 123], the Legendre polynomials satisfy (on $E = (-1, 1)$) the inequalities (8.9).

Example 8.5. (Simple Laguerre polynomials $L_n(t)$ [16, p. 201 and p. 213]).

$$e^x J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(t) \frac{x^n}{n!}, \quad t \in E = \mathbb{R},$$

where J_0 defined by (8.11). Since $J_0 \in \mathcal{L} - \mathcal{P}$, the Laguerre polynomials satisfy (8.9).

Example 8.6. (Hermite polynomials $H_n(t)$ [16, p. 187]).

$$e^{2xt} e^{-x^2} = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!}, \quad t \in E = \mathbb{R}.$$

Since $e^{2xt} e^{-x^2} \in \mathcal{L} - \mathcal{P}$ for all $t \in \mathbb{R}$, $\Delta_{n,p}(t) \geq 0$ for all $t \in \mathbb{R}$, $n, p = 0, 1, 2, \dots$

Example 8.7. (A special case when $u_n(t)$ in (8.7) is a transcendental entire function). For fixed $t \in E = \mathbb{R}$, let

$$F(x) = e^{x \cos t} \cos(x \sin t) = \sum_{n=0}^{\infty} \cos(nt) \frac{x^n}{n!}.$$

Then $F(x) \in \mathcal{L} - \mathcal{P}$ and the sequence $\{\cos(nt)\}_{n=0}^{\infty}$ satisfies (8.10) for all $t \in \mathbb{R}$.

As our last example we consider the Jensen polynomials (introduced in Section 1 (cf. Fact 1.4)) associated with an arbitrary function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}$. Now by considering the Cauchy product of the power series for e^x and $\varphi(xt)$, we find that

$$e^x \varphi(xt) = \sum_{n=0}^{\infty} g_n(t) \frac{x^n}{n!} \quad t \in \mathbb{R},$$

where

$$g_n(t) = \sum_{k=0}^n \binom{n}{k} \gamma_k t^k \quad n = 0, 1, 2, \dots$$

Since $\varphi(x) \in \mathcal{L} - \mathcal{P}$, for each fixed $t \in \mathbb{R}$, $e^x \varphi(xt) \in \mathcal{L} - \mathcal{P}$ and thus the Jensen polynomials satisfy (8.9) for all $t \in \mathbb{R}$.

9. OPEN PROBLEMS

9.1. The Hawai'i Conjecture and the Polar Laguerre Expression. In light of our results in Section 6, it seems plausible that the Hawai'i Conjecture is valid when expressed in terms of the polar derivative analog of the Laguerre inequality. First we recall that the Hawai'i Conjecture is stated as follows.

Conjecture 9.1 (The Hawai'i Conjecture). *If $p(x)$ is a real polynomial of degree $n \geq 2$ with $2d$ nonreal zeros, $0 \leq 2d \leq n$. Then*

$$Z_R(Q(x)) = Z_R\left(\left(\frac{p'(x)}{p(x)}\right)'\right) \leq 2d,$$

where $Z_R(Q)$ denotes the number of real zeros of the rational function Q , counting multiplicities.

Note that in the Hawai'i Conjecture $Q(x) = -\frac{L[p](x)}{(p(x))^2}$. Using this we state a polar derivative analog of the Hawai'i Conjecture as follows.

Conjecture 9.2. *If $p(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ is a real polynomial of degree $n \geq 2$ with $2d$ nonreal zeros, $0 \leq 2d \leq n$, then, for $\zeta \neq -\frac{a_{n-1}}{a_n}$,*

$$Z_R(Q_\zeta(x)) = Z_R\left(\frac{M_\zeta[p](x)}{(p(x))^2}\right) \leq 2d,$$

where $Z_R(Q_\zeta)$ denotes the number of real zeros of the rational function Q_ζ , counting multiplicity.

Remark 9.3. The denominator $(p(x))^2$ is necessary for the conjecture to hold. Consider, for example, $p(x) = x^4 - x^3 + x^2$. This polynomial has 2 nonreal zeros and a calculation shows that

$$M_0[p](x) = -x^4(x^2 + 2x - 2),$$

which has 6 real zeros. The division by $(p(x))^2$ cancels the multiple zero at 0, leaving 2 real zeros.

It is also possible that the Hawai'i Conjecture holds for functions in $\mathcal{L} - \mathcal{P}^*$.

Conjecture 9.4. *Suppose $\psi(x) \in \mathcal{L} - \mathcal{P}^*$ and $\psi(x)$ has $2d$ nonreal zeros, then*

$$Z_R\left(\left(\frac{\psi'(x)}{\psi(x)}\right)'\right) \leq 2d.$$

9.2. Krasikov's Discrete Laguerre Inequality. In [11] Krasikov introduces the discrete analogs of the Laguerre and Turán inequalities. According to Krasikov, these new inequalities have applications to classical orthogonal polynomials. In an attempt to make the discrete analogs sharper the author proposes the following conjecture.

Conjecture 9.5 (Krasikov's Conjecture). *Let $p(x) = \prod_{k=1}^n (x - \alpha_k) \in \mathcal{L} - \mathcal{P}$. Let $\mu_n(p) := \min_{1 \leq k \leq n-1} (\alpha_{k+1} - \alpha_k) = 1$. Then for all $x \in \mathbb{R}$,*

$$f_n(x, 1; p) := (n-1)[p(x+1) - p(x-1)]^2 - 4np(x)[p(x+1) - 2p(x) + p(x-1)] \geq 0.$$

In [5] Csordas proves that the following conjecture is equivalent to the above conjecture.

Conjecture 9.6. (An Equivalent Formulation of Krasikov's Conjecture [5]). *Let $p(x) = \prod_{k=1}^n (x - \alpha_k) \in \mathcal{L} - \mathcal{P}$. Suppose that $\mu_n(p) \geq t > 0$. Then for all $x \in \mathbb{R}$,*

$$f_n(x, t; p) := (n-1)[p(x+t) - p(x-t)]^2 - 4np(x)[p(x+t) - 2p(x) + p(x-t)] \geq 0.$$

It is also shown in [5] that it suffices to prove the conjecture for only one point and for all $p \in \mathcal{L} - \mathcal{P}$.

Proposition 9.7. [5] *Suppose that $f_n(0, 1; p) \geq 0$ for all $p(x) = \prod_{k=1}^n (x - \alpha_k) \in \mathcal{L} - \mathcal{P}$, then $f_n(x, 1; p) \geq 0$ for all $x \in \mathbb{R}$.*

9.3. Open Problems Involving the Polar Derivative. Since the usual derivative is the polar derivative with respect to $\zeta = \infty$, it is possible that there are polar derivative analogs of several classical theorems with ζ finite, but sufficiently large. This possibility also begs the question. Are these analogs more general than the classical results?

Conjecture 9.8. (The Polar Derivative Analog of the Gauss-Lucas Theorem). *If $p(z)$ is a complex polynomial, then all of the zeros of $T_\zeta[p](z)$ are located in the closed convex hull of the zeros of p for all $|\zeta| > \max |z_k|$, where z_k are the zeros of p .*

Conjecture 9.9. (The Polar Derivative Analog of Jensen's Theorem). *If $p(z)$ is a real polynomial, then the nonreal zeros of $T_\zeta[p](z)$ lie in or on some Jensen circle of p for all $|\zeta| > \max |z_k|$, where z_k are the zeros of p .*

Finally, it would be of interest to find a connection between the polar derivative and the metric conditions stated in Theorems 4.6 and 4.8 for a polynomial to have only real zeros. For example, is it true that a real polynomial $f(x)$ has only real zeros if and only if

$$\mu\left(\left\{x \in \mathbb{R} : \frac{T_\zeta[f](x)}{f(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda}k(\zeta),$$

where $k(\zeta)$ is some function of ζ and ζ is sufficiently large?

10. APPENDIX

10.1. Proof of Theorem 2.5 (Section 2.1). Motivation for Theorem 2.5 comes from the following example.

Example 10.1. *Using Mathematica we compute several extended Laguerre expressions for the function $g(x) = e^{-x^2}(x^2 + 10)$:*

$$\begin{aligned} L_1[g](x) &= 2e^{-x^2}(6 + x^2)(15 + x^2) \\ L_2[g](x) &= e^{-x^2}(161 + 44x^2 + 2x^4) \\ L_3[g](x) &= \frac{2}{3}e^{-x^2}(143 + 46 + 2x^4) \\ &\vdots \\ L_{15}[g](x) &= \frac{8e^{-x^2}(5 + 70x^2 + 2x^4)}{63851275} \\ L_{16}[g](x) &= \frac{2e^{-x^2}(x^2)(36 + x^2)}{63851275} \\ L_{17}[g](x) &= \frac{4e^{-x^2}(-2 + 37x^2 + x^4)}{10854718875} \end{aligned}$$

Thus, $L_k[g](x) \geq 0$ for all $x \in \mathbb{R}$ whenever $0 \leq k \leq 16$, but $L_{17}[g](0) < 0$.

Theorem 10.2. (Theorem 2.5, Section 2.1). *For each nonnegative integer n , there exists a function $g \in \mathcal{L} - \mathcal{P}^*$ such that $L_k[g](x) \geq 0$ for $0 \leq k \leq n$ and for all $x \in \mathbb{R}$, but $L_{n+1}[g](x) < 0$ for some $x \in \mathbb{R}$, where $L_k[g]$ denotes the extended Laguerre expression (cf. Section 2.1).*

Proof. Let $g(z) = e^{-z^2}(z^2 + a^2)$. Then $g \in \mathcal{L} - \mathcal{P}^*$ and

$$\begin{aligned} |g(x + iy)|^2 &= e^{-2x^2} e^{2y^2} [(x^2 + a^2)^2 + 2(x^2 - a^2)y^2 + y^4] \\ &= e^{-2x^2} \left\{ (x^2 + a^2)^2 + \left(\frac{2(x^2 + a^2)^2}{1!} + \frac{2(x^2 - a^2)}{0!} \right) y^2 \right. \\ &\quad \left. + \sum_{k=2}^{\infty} \left[(x^2 + a^2)^2 + k(x^2 - a^2) + \frac{1}{4}k(k-1) \right] \frac{2^k}{k!} y^{2k} \right\}. \end{aligned}$$

Thus, by Theorem 2.3, we have

$$\begin{aligned} L_0[g](x) &= e^{-2x^2}(x^2 + a^2)^2, \\ L_1[g](x) &= e^{-2x^2}[2(x^2 + a^2)^2 + 2(x^2 - a^2)], \end{aligned}$$

and for $k \geq 2$

$$L_k[g](x) = e^{-2x^2} \left[(x^2 + a^2)^2 + k(x^2 - a^2) \frac{1}{4}k(k-1) \right] \frac{2^k}{k!}.$$

For simplicity of notation let $b = a^2$. Then $g(z) = e^{-z^2}(z^2 + b)$. Thus,

$$\begin{aligned} L_0[g](x) &= e^{-2x^2}(x^2 + b)^2 = e^{-2x^2}[x^4 + 2bx^2 + b^2], \\ L_1[g](x) &= e^{-2x^2}[2(x^2 + b)^2 + 2(x^2 - b)] \\ &= 2e^{-2x^2}[x^4 + x^2(2b + 1) + b^2 - b], \end{aligned}$$

and for $k \geq 2$

$$\begin{aligned} L_k[g](x) &= e^{-2x^2} \left[(x^2 + b)^2 + k(x^2 - b) + \frac{1}{4}k(k-1) \right] \frac{2^k}{k!} \\ &= e^{-2x^2} \frac{2^k}{k!} \left[x^4 + x^2(2b + k) + q_k(b) \right], \end{aligned}$$

where $q_k(b) = b^2 - kb + \frac{1}{4}k(k-1)$. Fix a nonnegative integer n . Let $b = b_n = \frac{1}{2}(n + \sqrt{n})$. Then

$$q_k(b) = \frac{1}{4}(n-k)(n+1-k+2\sqrt{n})$$

and whence

$$(10.1) \quad q_k(b) \geq 0 \quad (0 \leq k \leq n) \quad \text{and} \quad q_{n+1}(b) < 0.$$

Since

$$\operatorname{sgn}(L_k[g](x)) = \operatorname{sgn}[x^4 + x^2(2b + k) + q_k(b)],$$

$L_k[g](x) \geq 0$ for all $x \in \mathbb{R}$, whenever $q_k(b) \geq 0$. Therefore it follows from (10.1) that

$$L_k[g](x) \geq 0 \quad \text{for all } x \in \mathbb{R} \quad \text{and} \quad 0 \leq k \leq n$$

and

$$L_{n+1}[g](0) < 0.$$

□

10.2. Calculation of the Polar Derivative Laguerre Expression. If

$$T_\zeta[f](x) = nf(x) + (\zeta - x)f'(x),$$

then

$$(T_\zeta[f](x))^2 = n^2(f(x))^2 + 2n(\zeta - x)f'(x)f(x) + (\zeta - x)^2(f'(x))^2$$

and

$$\begin{aligned} T_\zeta T_\zeta[f](x) &= (n-1)T_\zeta[f](x) + (\zeta - x)(T_\zeta[f](x))' \\ &= (n-1)(nf(x) + (\zeta - x)f'(x)) + \\ &\quad (\zeta - x)[nf'(x) + (\zeta - x)f''(x) - f'(x)] \\ &= n^2f(x) + n(\zeta - x)f'(x) - nf(x) - (\zeta - x)f'(x) \\ &\quad n(\zeta - x)f'(x) + (\zeta - x)^2f''(x) - (\zeta - x)f'(x). \end{aligned}$$

Moreover,

$$\begin{aligned} M_\zeta[f](x) &= (T_\zeta[f](x))^2 - f(x)T_\zeta T_\zeta[f](x) \\ &= n^2(f(x))^2 + 2n(\zeta - x)f'(x)f(x) + (\zeta - x)^2(f'(x))^2 \\ &\quad - n^2(f(x))^2 - 2n(\zeta - x)f'(x)f(x) + n(f(x))^2 + (\zeta - x)f'(x)f(x) \\ &\quad - (\zeta - x)^2f''(x)f(x) + (\zeta - x)f'(x)f(x) \\ &= (\zeta - x)^2[(f'(x))^2 - f''(x)f(x)] + n(f(x))^2 + 2(\zeta - x)f'(x)f(x) \\ &= (\zeta - x)^2L[f](x) + f(x)T_\zeta[f](x) + (\zeta - x)f'(x)f(x). \end{aligned}$$

10.3. Composition Theorems. By repeated application of Laguerre's Separation Theorem, we now prove Grace's Apolarity Theorem. This fundamental result relating the relative location of the zeros of two apolar polynomials, while remarkable for its lack of intuitive content, has far-reaching consequences. One such consequence is the Malo-Schur-Szegö Composition Theorem stated and proved below (Theorem 10.9).

Definition 10.3. Two polynomials

$$f(x) := \sum_{k=0}^n a_k \binom{n}{k} x^k, \quad a_n \neq 0$$

and

$$g(x) := \sum_{k=0}^n b_k \binom{n}{k} x^k, \quad b_n \neq 0$$

are called *apolar* if

$$\sum_{k=0}^n (-1)^k a_k b_{n-k} \binom{n}{k} = 0.$$

Theorem 10.4. (Grace's Apolarity Theorem [1, p. 23]). *If f and g are two apolar polynomials and if f has all of its zeros in a disk D (D can be open or closed), then g has at least one zero in the disk D .*

Proof. Suppose that $\alpha_1, \alpha_2, \dots, \alpha_n$ are the zeros of f and that $\beta_1, \beta_2, \dots, \beta_n$ are the zeros of g . Suppose that all of the zeros of f lie in the disk D and that all of the zeros of g lie outside of the disk D . Consider the k^{th} polar derivative of f with respect to the point β_k . For $k = 1$

$$T_{\beta_1}[f](x) := f_1(x) = n f(x) + (\beta_1 - x) f'(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + \beta_1 a_{k+1}) x^k,$$

and for $k = 2, 3, \dots, n$,

$$T_{\beta_k} T_{\beta_{k-1}} \cdots T_{\beta_1}[f](x) := f_k(x) = (n - k + 1) f_{k-1}(x) + (\beta_k - x) f'_{k-1}(x).$$

By Laguerre's Separation Theorem, each f_k has its zeros in D , since each β_k is outside of D .

Then

$$\begin{aligned} f_1(x) &= n \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + \beta_1 a_{k+1}) x^k \\ f_2(x) &= n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} (a_k + (\beta_1 + \beta_2) a_{k+1} + \beta_1 \beta_2 a_{k+2}) x^k \\ &\vdots \\ f_{n-1}(x) &= n! \sum_{k=0}^1 \binom{1}{k} \left[\sum_{j=0}^{n-1} a_{k+j} \sigma_j \right] x^k, \end{aligned}$$

where σ_j is the j^{th} elementary symmetric function of $\beta_1, \beta_2, \dots, \beta_{n-1}$. It now follows that

$$\begin{aligned}
 f_{n-1}(x) &= n! \sum_{k=0}^1 \binom{1}{k} \left[\sum_{j=0}^{n-1} a_{k+j} \sigma_j \right] x^k \\
 &= n! \left(\sum_{j=0}^{n-1} a_j \sigma_j + \sum_{j=0}^{n-1} a_{j+1} \sigma_j x \right) \\
 &= n! \left(a_0 \sigma_0 + a_n \sigma_{n-1} x + \sum_{j=1}^{n-1} a_j \sigma_j + \sum_{j=1}^{n-1} a_j \sigma_{j-1} x \right) \\
 &= n! \left(a_0 \sigma_0 + a_n \sigma_{n-1} x + \sum_{j=0}^{n-1} (\sigma_j + \sigma_{j-1} x) a_j \right).
 \end{aligned}$$

By writing $g(x)$ as a product, we have

$$\begin{aligned}
 g(x) &= \sum_{k=0}^n \binom{n}{k} b_k x^k = \prod_{k=1}^n b_n (x - \beta_k) \\
 &= \sum_{k=0}^n b_n (-1)^k \Sigma_k x^{n-k} \\
 &= \sum_{k=0}^n b_n (-1)^{n-k} \Sigma_{n-k} x^k,
 \end{aligned}$$

where Σ_k is the k^{th} elementary symmetric function of $\beta_1, \beta_2, \dots, \beta_n$. Equating the coefficients of x^k we have,

$$\Sigma_{n-k} = \frac{(-1)^{n-k} b_k \binom{n}{k}}{b_n}$$

and relabelling the indices yields,

$$\Sigma_k = \frac{(-1)^k b_{n-k} \binom{n}{n-k}}{b_n} = \frac{(-1)^k b_{n-k} \binom{n}{k}}{b_n}.$$

Evaluating the $(n-1)^{\text{st}}$ polar derivative of f at $x = \beta_n$, we have

$$f_{n-1}(\beta_n) = n! \left(a_0 \sigma_0 + a_n \sigma_{n-1} \beta_n + \sum_{j=0}^{n-1} (\sigma_j + \sigma_{j-1} \beta_n) a_j \right).$$

But

$$\begin{aligned}
 \sigma_0 &= \Sigma_0 \\
 \sigma_{n-1} \beta_n &= \Sigma_n \\
 &\text{and} \\
 \sigma_j + \sigma_{j-1} \beta_n &= \Sigma_j.
 \end{aligned}$$

Thus,

$$\begin{aligned} f_{n-1}(\beta_n) &= n! \left(a_0 \sigma_0 + a_n \sigma_{n-1} \beta_n + \sum_{j=0}^{n-1} (\sigma_j + \sigma_{j-1} \beta_n) a_j \right) \\ &= n! \sum_{j=0}^n a_j \Sigma_j. \end{aligned}$$

But substituting our equation for Σ_k we get,

$$n! \sum_{j=0}^n a_j \Sigma_j = \frac{n!}{b_n} \sum_{j=0}^n a_j b_{n-j} (-1)^j \binom{n}{j}.$$

Therefore,

$$f_{n-1}(\beta_n) = \frac{n!}{b_n} \left[\binom{n}{0} a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \cdots + (-1)^n \binom{n}{n} a_n b_0 \right] = 0,$$

by apolarity. Therefore, $\beta_n \in D$, which contradicts the choice of the β_k 's. \square

Before proving the Malo-Schur-Szegö Composition Theorem, we recall several fundamental results from the theory of distribution of zeros of polynomials.

As in Remark 6.10, we define the operator, \star , acting on a polynomial p of degree n , by

$$p^\star(x) = x^n p\left(\frac{1}{x}\right).$$

We remark that simple examples show that if the polynomial $p(x)$ vanishes at the origin, then $Z_R(p^\star) \neq Z_R(p)$.

Lemma 10.5. *The operator \star preserves the number of nonreal zeros of a polynomial; that is,*

$$(10.2) \quad Z_C(p^\star) = Z_C(p),$$

where $p(x)$ is a real polynomial.

Theorem 10.6. ([15, Problem 60, Part V, Chapter 1]). *Let $\nu \in \mathbb{Z}^+$ with $0 \leq \nu \leq n$. Let $f(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ be a real polynomial. Define $g(x) = \sum_{k=0}^n \binom{\nu}{k} a_k x^k$. Then*

$$Z_C(g) \leq Z_C(f).$$

Theorem 10.7. (Hermite-Poulain Theorem). *Suppose that $f(x)$ is a real polynomial and $g(x) = \sum_{k=0}^n a_k x^k \in \mathcal{L} - \mathcal{P}$. Then*

$$h(x) = g(D)f(x) = \sum_{k=0}^n a_k f^{(k)}(x), \quad (D = \frac{d}{dx})$$

has no more nonreal zeros than f ; i.e.,

$$Z_C(f) \leq Z_C(h).$$

Corollary 10.8. *If $p(x) = \sum_{k=0}^n a_k x^k \in \mathcal{L} - \mathcal{P}$, then*

$$(10.3) \quad q(x) = \sum_{k=0}^n a_k m(m-1)(m-2) \cdots (m-k+1) x^{m-k} \in \mathcal{L} - \mathcal{P}$$

and

$$(10.4) \quad r(x) = \sum_{k=0}^n \frac{a_k}{k!} x^k \in \mathcal{L} - \mathcal{P}.$$

Proof. Apply the Hermite-Poulain Theorem (Theorem 10.7) to $p(x)$ with $f(x) = x^m$. Then

$$q_1(x) = \sum_{k=0}^n a_k m(m-1)(m-2) \cdots (m-k+1) x^{m-k} \in \mathcal{L} - \mathcal{P}.$$

By Lemma 10.5

$$q(x) = q_1^*(x) = \sum_{k=0}^n a_k m(m-1) \cdots (m-k+1) x^k \in \mathcal{L} - \mathcal{P}.$$

Now apply the Hermite-Poulain Theorem (Theorem 10.7) with $f(x) = x^n$ to the polynomial

$$r^*(x) = \sum_{k=0}^n a_k x^{n-k} = \sum_{k=0}^n a_{n-k} x^k \in \mathcal{L} - \mathcal{P}.$$

It follows that

$$\begin{aligned} r_1(x) &= \sum_{k=0}^n a_{n-k} (x^n)^{(k)} \\ &= \sum_{k=0}^n a_{n-k} \frac{n!}{(n-k)!} x^{n-k} \\ &= n! \sum_{k=0}^n \frac{a_k}{k!} x^k = n! r(x) \in \mathcal{L} - \mathcal{P}. \end{aligned}$$

□

We now state and prove the Malo-Schur-Szegő Composition Theorem.

Theorem 10.9. (The Malo-Schur-Szegő Composition Theorem [2]). *Let*

$$A(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k \quad \text{and} \quad B(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k$$

and set

$$C(z) = \sum_{k=0}^n \binom{n}{k} a_k b_k z^k.$$

- (1) (Szegő) *If all the zeros of $A(z)$ line in a circular region K (K can be open or closed), and if $\beta_1, \beta_2, \dots, \beta_n$ are the zeros of $B(z)$, then every zero of $C(z)$ is of the form $\gamma = -\alpha\beta_j$, for some j , $1 \leq j \leq n$, and some $\alpha \in K$.*
- (2) (Schur) *If all the zeros of $A(z)$ lie in a convex region K containing the origin and if the zeros of $B(z)$ lie in the interval $(-1, 0)$, then the zeros of $C(z)$ also lie in K .*
- (3) *If the zeros of $A(z)$ lie in the interval $(-a, a)$ and if the zeros of $B(z)$ lie in the interval $(-b, 0)$ (or in $(0, b)$), where $a, b > 0$, then the zeros of $C(z)$ lie in $(-ab, ab)$.*

- (4) (*Malo, Schur*) If the zeros of $p(z) = \sum_{k=0}^{\mu} a_k z^k$ are all real and if the zeros of $q(z) = \sum_{k=0}^{\nu} b_k z^k$ are all real and of the same sign, then the zeros of the polynomials $h(z) = \sum_{k=0}^m k! a_k b_k z^k$ and $f(z) = \sum_{k=0}^m a_k b_k z^k$ are also all real, where $m = \min(\mu, \nu)$.

Proof.

- (1) (Szegő) Suppose that $A(z)$ has all of its zeros in a circular region K . Let $\beta_1, \beta_2, \dots, \beta_n$ be the zeros of $B(z)$. Suppose that γ is an arbitrary zero of $C(z)$. Then

$$C(\gamma) = \sum_{k=0}^n \binom{n}{k} a_k b_k \gamma^k = 0.$$

Define

$$R(z) := \sum_{k=0}^n \binom{n}{k} (-1)^k b_k \gamma^k z^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} b_{n-k} \gamma^{n-k} z^k.$$

Note that $R(z)$ is apolar to $A(z)$,

$$\sum_{k=0}^n \binom{n}{k} (-1)^k a_k (-1)^k b_k \gamma^k = \sum_{k=0}^n a_k b_k \gamma^k = 0.$$

Thus, by Grace's Theorem $R(z)$ has a zero $\alpha \in K$. Since,

$$\begin{aligned} z^n B\left(\frac{-\gamma}{z}\right) &= z^n \sum_{k=0}^n \binom{n}{k} b_k (-1)^k \gamma^k z^{-k} \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k b_k \gamma^k z^{n-k} = R(z) \end{aligned}$$

it follows that

$$R(\alpha) = \alpha^n B\left(\frac{-\gamma}{\alpha}\right) = 0.$$

Note that $b_n \neq 0$ (otherwise there would not be n zeros of $B(z)$), so $R(z)$ does not have a zero at 0 and $\alpha \neq 0$. Thus, $\frac{-\gamma}{\alpha} = \beta_k$ for some k ; i.e., $\gamma = -\alpha\beta_k$. Since γ was an arbitrary zero of $C(z)$ we know that all zeros of $C(z)$ are of the form $\gamma = -\alpha\beta_k$.

- (2) (Schur) Let K be the convex hull containing the origin and all the zeros of $A(z)$. Then each side of K determines a line which is the boundary of a half plane. Since all of the zeros lie on one side of the half plane by (1) there exists an α on that side of the half plane such that $\gamma = -\alpha\beta_k$ where γ is a zero of $C(z)$. Since $|\beta_k| < 1$, we have $|\gamma| < |\alpha|$; i.e., γ is closer to the origin than α . By considering all half planes that are determined by the sides of K , we conclude that all the zeros of $C(z)$ lie in K .
- (3) Since all the zeros of $A(z)$ lie in $(-a, a)$ we consider the upper half plane so that the boundary the real line. By (1) any zero, γ , of $C(z)$ can be written $\gamma = -\alpha\beta_k$ where α is in the same half plane as the zeros of $A(z)$. By considering the lower half plane we conclude that $\alpha \in \mathbb{R}$. Thus, $\gamma = -\alpha\beta_k \in \mathbb{R}$. Since $0 < |\beta_k| < b$ we have, $|\gamma| = |\alpha\beta_k| < |\alpha|b$. If we now consider the circular region $\{z : |z| < a\}$, by (1) $\alpha \in (-a, a)$ and $|\alpha| < a$. Therefore, $|\gamma| < |\alpha|b < ab$, and any zero of $C(z)$ lies in the interval $(-ab, ab)$.

(4) (Malo, Schur) Let

$$p(x) = \sum_{k=0}^{\mu} a_k x^k = \sum_{k=0}^n \binom{\mu}{k} \alpha_k x^k$$

and

$$q(x) = \sum_{k=0}^{\nu} b_k x^k = \sum_{k=0}^{\nu} \binom{\nu}{k} \beta_k x^k$$

have only real zeros and suppose all the zeros of q are of the same sign. Suppose that $m = \mu = \max(\mu, \nu)$ (the proof when $m = \nu$ is the same). Then by Theorem 10.6 we know that

$$\tilde{q}(x) = \sum_{k=0}^m \binom{m}{k} \beta_k x^k \in \mathcal{L} - \mathcal{P}.$$

We also know from the proof of Theorem 10.6 that all the zeros of \tilde{q} will still be of the same sign. By Part (3) of Theorem 10.9, we know that

$$\begin{aligned} C(x) &= \sum_{k=0}^m \binom{m}{k} \alpha_k \beta_k x^k \\ &= \sum_{k=0}^m a_k \beta_k x^k \\ &= \sum_{k=0}^m \frac{k!(\nu - k)! a_k b_k}{\nu!} x^k \in \mathcal{L} - \mathcal{P}. \end{aligned}$$

By Corollary 10.8 (10.3),

$$h(x) = \sum_{k=0}^m k! a_k b_k x^k \in \mathcal{L} - \mathcal{P}.$$

Furthermore, by Corollary 10.8 (10.4), it follows that

$$f(x) = \sum_{k=0}^m a_k b_k x^k \in \mathcal{L} - \mathcal{P}.$$

□

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